A Diameter-Constrained Network Reliability model to determine the Probability that a Communication Network meets Delay Constraints

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Abstract: - In this paper we provide a summary of results and applications pertaining a Diameter-constrained Network reliability model. Classical network reliability models measure the probability that there exist end-to-end paths between network nodes, not taking into account the length of these paths. For many applications this is inadequate because the connection will only be established or attain the required quality if the distance between the connecting nodes does not exceed a given value.
The *Diameter-constrained reliability* of a network (**DCR**) introduced recently considers not only the underlying topology, but also imposes a bound on the diameter, which is the maximum distance between the nodes of the network.
We present a synopsis of the known results and applications of the DCR for networks that can either be modeled by directed as well as undirected graphs.
Moreover important combinatorial and computational properties of this reliability measure are discussed. As the DCR subsumes the classical reliability measure (i.e., where no distance constraints are imposed on the paths connecting the nodes), as a by-product we prove well-known classical results.

Key–Words: - Reliability, networks, diameter, computational complexity, delay constraints, domination.

1 Introduction

The purpose of this paper is to present a summary of results obtained hitherto regarding a network reliability model, the *Diameter-constrained reliability* (**DCR**) originally introduced in 2001 (see [6, 19]).
Even though the proofs of the results appear elsewhere, here we present a brief explanation of the techniques used to prove them.

The components of a communication network (e.g. nodes, communication links) may be subject to random failures. Failures may arise from natural catastrophes (e.g. hurricanes), component wearout, or action of intentional enemies.

A communication network can be modeled by a graph (or digraph) $G = (V, E)$ where V and E are the set of vertices and edges (arcs) respectively of G. Moreover the probabilities of failure of the network components could be represented by assigning probabilities of failure to the vertices and/or edges (arcs) of its underlying graph (digraph).

A widely used probabilistic model is the one where the edges (arcs) fail randomly and independently with known probabilities, and where the vertices are always operational.

A *path (dipath)* P between two vertices u and v is a sequence of distinct vertices $< u_1 = u, u_2, \ldots, u_r = v >$ where (u_i, u_{i+1}) is an edge (arc) of G, $1 \leq i \leq r-1$. Moreover the length of P is $r-1$. We also represent a *cycle (dicycle)* as a sequence of vertices $< u_1, u_2, \ldots, u_{r+1} = u_1 >$, where $(u_j, u_{j+1}) \in E$, and where all the vertices are distinct with the exception of the first and last vertices of the sequence, and we say that a graph (digraph) is *cyclic* if it contains a cycle (dicycle), otherwise it is *acyclic*.

Let $G = (V, E)$ be a undirected graph with a distinguished set $K \subseteq V$. We define the K-diameter of G as the maximum distance between any pair of vertices of K. If the edges fail randomly and independently with known probabilities, in [6] and [19] the *Diameter-constrained K-terminal reliability* of G, $R_K(G, D)$, was defined as the probability that surviving edges span a subgraph whose K-diameter does not exceed D, or equivalently, as the probability that for each pair of vertices $\{u, v\} \subseteq K$, there exists an operating path between u and v of at most D edges.

As real networks are subject to failures, the diameter-constrained reliability can be useful in different contexts. For example, this measure gives an indicator of the suitability of an existing network topology to support good quality voice over IP applications between a pair of terminals. In the case of a videoconference, we take K to be the set of the participating nodes, and the Diameter-constrained reliability gives the probability that we can find short enough paths between all of them. Another potential case of interest are a number of protocols which, in order to avoid congestion by looping data, assign a timeout date or a maximum number of hops to each data packet, to control information. That is the case for some peer-to-peer (P2P) networks, such as Freenet, Gnutella or others [9, 13, 18, 22, 28], in which a fixed maximum number of hops are allowed for communication between nodes. In these cases, the diameter-constrained unreliability (the complement to one of the reliability) gives the probability that, due to failed links, there are some nodes of the network which are not mutually reachable using these protocols.

Similarly a network can be modeled by a digraph $G = (V, E)$ where V and E are the set of vertices and arcs respectively of G. In particular we study the case where we want to model the connection between a source-vertex s and a set of terminal vertices K (for example to model a video multi-cast application). We denote the s, K-diameter as the maximum distance between s and any of the vertices of K. Then the *Diameter-constrained s, K-terminal reliability* of a network G, $R_{s,K}(G, D)$, is defined as the probability that the surviving arcs span a subgraph whose s, K-diameter does not exceed D ([6, 19]).

The Diameter-constrained network reliability is a special case of coherent system models, where the *domination* invariant has played an important role, both theoretically and for developing of efficient algorithms for the reliability computation.

In Section 2 we introduce some basic notation and definitions for the Diameter-constrained K-terminal reliability of an undirected graph G, $R_K(G, D)$, and we present $R_K(G, D)$ as a generalization of the classical reliability measure $R_K(G)$, allowing us to conclude that in general the complexity of evaluating $R_K(G, D)$ is NP-hard. In addition, we show that calculation of the DCR remains an NP-hard problem, even for a fixed number of terminal vertices and for fixed diameter bound D. We also present a backtracking algorithm to compute the DCR if an undirected topology.

In Section 3 we discuss the Diameter-constrained s, K-terminal reliability of a digraph G, $R_{s,K}(G, D)$, and we completely characterize the domination of diameter-constrained network models, giving a simple rule for computing its value: if the digraph either has an irrelevant arc, includes a directed cycle or includes a dipath from s to a node in K longer than D, its domination is 0; otherwise, its domination is -1 to the power $|E| - |V| + 1$. In particular this characterization yields the classical Source-to-K-terminal reliability domination obtained by Satyanarayana [24]. Based on these theoretical results, we present an algorithm for computing the reliability.

2 Undirected model

In this section we present several results pertaining the Diameter-constrained K-terminal reliability of an undirected graph G, $R_K(G, D)$. We first introduce basic notation and definitions that will be used in the sequel:

- Let $G = (V, E, \mathcal{P}(E))$ be a probabilistic graph with a distinguished set $K \subseteq V$, $|K| \geq 2$, and $D \in Z^+$, with $1 \leq D \leq n - 1$, where $n = |V|$, and where $\mathcal{P} : E \mapsto [0, 1]$ are the operational probabilities of the set of edges E. For ease of notation, we represent the operational probability of an edge $e \in E$ as $p(e) = 1 - q(e)$ ($q(e)$ is the probability of failure).
- Let the sample space Ω represent the set of all possible subsets of E, corresponding to sets of operational edges (i.e. $\Omega = 2^E$).
- Under the assumption of independent edge failures, each $H \in \Omega$ has occurrence probability

$$P(H) = \prod_{e \in H} p(e) \prod_{e \notin H} q(e).$$

- $H \in \Omega$ is a *pathset* or *operating state* if H spans a subgraph whose K-diameter is at most D.
- Let $\mathcal{O}_K^D(E) = \{H \in \Omega : H \text{ is a pathset}\}$.
- An operating state H of $\mathcal{O}_K^D(E)$ is called a *minpath* if $H - \{e_i\} \notin \mathcal{O}_K^D(E)$, for all $e_i \in H$.
- $H \in \Omega$ is a *failure state* if H spans a subgraph whose K-diameter is greater than D (if H spans a subgraph where two vertices of K belong to different connected components, then its K-diameter is infinite).
- Let $\overline{\mathcal{O}}_K^D(E) = \{H \in \Omega : H \text{ is a failure state}\}$.

From the definition of $R_K(G, D)$ and the previous definitions one gets

$$R_K(G, D) = \sum_{H \in \mathcal{O}_K^D(E)} \prod_{e \in H} p(e) \prod_{e \notin H} q(e). \qquad (1)$$

Similarly equation (1) can be rewritten in terms of the failure states:

$$R_K(G,D) = 1 - Q_K(G,D) = 1 - \sum_{H \in \overline{\mathcal{O}}_K^D(E)} \prod_{e \in H} p(e) \prod_{e \notin H} q(e). \quad (2)$$

2.1 Computational Complexity and DCR as a generalization of the classical reliability model

A widely used probabilistic measure (see [5, 10, 23, 26]) is the classical *K-terminal reliability* of a graph G, $R_K(G)$, defined on the same probabilistic model (i.e., edges fail randomly and independently with known probabilities and the vertices are always operational). The measure $R_K(G)$ is the probability that for every pair of vertices $u, v \in K$, there exists an operating path between u and v. In this case there are not length restrictions of the paths joining the vertices of K, and by noting that the maximum length of a path joining a pair of vertices is of at most $n-1$ edges, where n is the number of vertices of G, then

$$R_K(G) = R_K(G, n-1). \quad (3)$$

This generalization of the classical reliability parameter allows us to reflect more stringent performance objectives by restricting the maximum length of a path in a network.

Let $G = (V,E)$ and K be a set of terminal vertices of G. For the classical reliability measure, computations of the K-terminal reliability (see [21]), and the specific cases when $|K| = 2$ (see [27]), and $K = V$ (see [16, 20]) , were shown to be NP-hard. From these results and the fact that $R_K(G) = R_K(G, n-1)$, $R_{\{s,t\}}(G) = R_{\{s,t\}}(G, n-1)$, and $R_V(G) = R_V(G, n-1)$, where n is the number of vertices of the graph G, and by restricting $D = n-1$, then

Theorem 1 *For the Diameter-constrained reliability, the computational complexity of computing $R_K(G,D)$, $R_{\{s,t\}}(G,D)$, and $R_V(G,D)$ is NP-hard.*

Even though is very unlikely that $R_K(G,D)$ can be evaluated efficiently, we can not preclude that is the case when fixed values of the diameter parameter D are under consideration. We address this question in Section 2.2 and Section 2.3.

2.2 Evaluating $R_K(G,D)$ when $|K| = 2$, and when D is a constant value

In this section we establish the computational complexity of computing $R_K(G,D)$, when K is composed of two terminal vertices s and t, and for fixed values of diameter parameter D.

In [19], an efficient formulation was given for the evaluation of Diameter-constrained Two-terminal Reliability of a network when terminals s and t should be connected by operating paths of at most two edges (i.e. $D = 2$).

Theorem 2 *Let $G = (V,E)$ be a simple graph where each edge $(u,v) \in E$ operates independently with probability $p_{(u,v)}$, and let $\mathcal{N}_{\{s,t\}} = \{u_1, u_2, \ldots, u_l\}$ be the common neighborhood of terminal nodes s and t, then*

$$R_{\{s,t\}}(G,2) = \begin{cases} 1 - R' & : \quad (s,t) \notin E \\ 1 - (1 - p_{(s,t)})R' & : \quad (s,t) \in E \end{cases}$$

where

$$R' = \prod_{i=1}^{l} (1 - p_{(s,u_i)} p_{(u_i,t)}).$$

Even though $R_{\{s,t\}}(G,2)$ can be computed in time linear on the number of vertices of G, we next show that the complexity of evaluating $R_{\{s,t\}}(G,D)$, for fixed values of D, is NP-hard.

For ease of notation instead of representing a state (operational or failure) as a set of edges of a graph G, we represent it as a subgraph of G spanned by this set.

Theorem 3 *Evaluating $R_{\{s,t\}}(G,D)$, for fixed $D \geq 3$, is NP-hard.*

An instance of the Bipartite Vertex Cover consists of a bipartite graph $G = (V,E)$; let X and Y be the classes in the bipartition of V. A vertex cover is a set of vertices $C = C_X \cup C_Y$, $C_X \subseteq X$ and $C_Y \subseteq Y$, such that every edge of E has at least one end-point in C. The problem of counting the number of vertex covers of a bipartite graph was shown to be $\#\mathcal{P}$-complete by Provan and Ball [20]. In [7] a graph

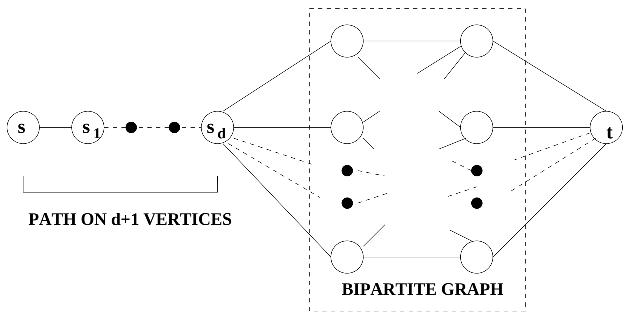

Figure 1: Graph G' constructed from bipartite G, and constant $D = d + 3$.

G' with terminal vertices $K = \{s,t\}$ and diameter $D = d + 3$ is constructed from a bipartite G and it is shown that counting the number of vertex covers of a bipartite graph is polynomial time Turing reducible to counting the number of failure states of G' (see Figure 1).

2.3 Evaluating $R_K(G, D)$ for any fixed number of terminal vertices K, and when D is a constant value, $D \geq 3$

As in the previous section, it can be shown that the complexity of calculating $R_K(G, D)$ is NP-hard, even for a fixed number of terminal vertices and for fixed diameter bound D, $D \geq 3$:

Theorem 4 *Evaluating $R_K(G, D)$ for a fixed number of terminal vertices, and for fixed D, $D \geq 3$, is NP-hard.*

To prove Theorem 4 (see [7]) a graph G'' is constructed by adding $|K| - 2$ terminal vertices to graph G' mentioned in the previous section (i.e., G'' has $|K|$ terminal vertices). Moreover we make each of these new terminal vertices adjacent to s, and to the terminal vertex t by a path of length D (see Figure 2).

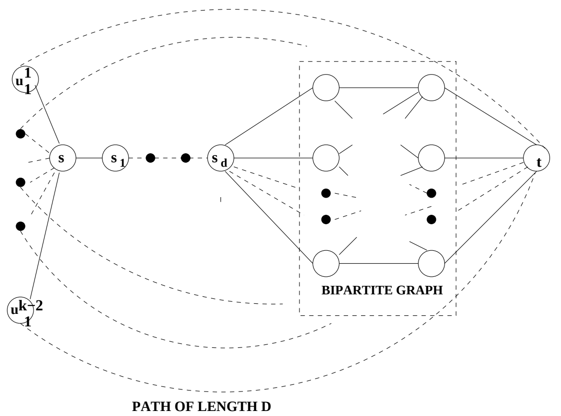

Figure 2: Graph G'' constructed from bipartite graph G.

As in the proof of Theorem 3, is easy to show that the number of vertex covers of the bipartite G is polynomial time Turing reducible to counting the number of failure states of G''.

2.4 Algorithm to compute the DCR

The simplest way to exactly evaluate $R_K(G, D)$ for a graph $G = (V, E)$ with diameter bound D is to enumerate all possible states (i.e., subgraphs) of E, and determine the ones whose K-diameter is at most D (i.e., the pathsets), and then sum the pathsets probabilities of being operative (see Equation 1).

The K-diameter of a state can be determined by application of Floyd algorithm [11] for determining the distance between every pair of vertices of the graph, and a state, is a pathset, if the distance between every pair of vertices of the terminal set K is D or less.

The drawback of this approach is that there are $2^{|E|}$ possible states to be analized, however, by introducing a backtracking technique, many states could be avoided.

Starting with the original graph G, at a particular state we determine if it is a pathset by application of Floyd's algorithm (of complexity $O(|V|^3)$), and then making recursive calls of states with one edge less. Once we have reached a state whose K-diameter is more than D, it is not necessary to consider any of its subgraphs since any subgraph of this state has also K-diameter exceeding D.

Bactracking Algorithm

Input: Graph $G = (V, E)$, edges prob. of survival function $P : E \to [0, 1]$, terminal set $K \subseteq V$, and $D \in Z^+$, $1 \leq D \leq |V| - 1$.
Output: Diameter-Constrained K-terminal Reliabilty $R_K(G, D)$.
Global variables:

- edges binary vector $\bar{e} = [m_1, m_2, \ldots, m_{|E|}]$,$m_i = 1, 1 \leq i \leq |E|$;
- edges probabilities vector $\bar{p} = [p_1, p_2, \ldots, p_{|E|}]$, $p_i = P(e_i), e_i \in E$;
- distance matrix $[d_{i,j}]_{nXn}$, $n = |V|$;
- reliability $R = 0$;

1. Call Backtrack (0);
2. Print R;

Backtrack (int i)
1. Construct distance matrix $[d_{i,j}]$ from vector $\bar{e}$;
2. Call Floyd($[d_{i,j}]$); determine if $\bar{e}$ is a pathset
3. If ($\bar{e}$ is a pathset)
    3.1 Let $R = R + \prod_{m_i=1} p_i \prod_{m_i=0}(1 - p_i)$;
    3.2 For (int $j = i + 1, j <= |E|; j + +$)
        3.2.1 Let $m_j = 0$ in $\bar{e}$; delete edge e_j
        3.2.2 Backtrack (j);
        3.2.3 Let $m_j = 1$ in $\bar{e}$; add edge back

In step 1 of the **Backtrack** procedure, the distance matrix $[d_{i,j}]$ is constructed from the binary vector $\bar{e}$ by letting $d_{i,j} = 1$ if $(i, j) \in \bar{e}$, $d_{i,j} = \infty$ if $(i, j) \notin \bar{e}$, and $d_{i,i} = 0$. A state is determined to be a pathset (step 3) if the distance returned between every pair of vertices of the terminal set K is D or less (Floyd's procedure).

3 Directed Model

In this section we present several results pertaining the Diameter-constrained s, K-terminal reliability of a digraph G, $R_{s,K}(G, D)$. We first introduce basic notation and definitions that will be used in the sequel:

- We denominate $ind_G(v)$ (or indegree of vertex v) the number of arcs directed into a vertex v of the digraph G.
- Let $outd_G(v)$ (or outdegree of vertex v) be the number of arcs emanating from a vertex v of the digraph G.
- Let $G = (V, E, \mathcal{P}(E))$ be a probabilistic digraph with a distinguished set $K \subseteq V$, vertex $s \in K$, and $D \in Z^+$, with $D \geq 0$, where $n = |V|$, and where $\mathcal{P} : E \mapsto [0, 1]$ are the operational probabilities of the arcs in set E. We represent the operational probability of an arc (or arc reliability) $x \in E$ as $p(x) = 1 - q(x)$ ($q(x)$ is the probability of failure).
- Let the sample space Ω represent the set of all possible subsets of E, corresponding to sets of operational arcs (i.e. $\Omega = 2^E$).
- Under independent failures assumption each $H \in \Omega$ has occurrence probability
$$P(H) = \prod_{x \in H} p(x) \prod_{x \notin H} q(x).$$
- $H \in \Omega$ is a *pathset* or *operating state* if H spans a subgraph whose s, K-diameter is at most D.
- Let $\mathcal{O}^D_{s,K}(E) = \{H \in \Omega : H \text{ is a pathset }\}$.
- An operating state H of $\mathcal{O}^D_{s,K}(E)$ is called a *minpath* if $H - \{x_i\} \notin \mathcal{O}^D_{s,K}(E)$, for all $x_i \in H$ (i.e, a minpath is a minimal operating state).
- A K-tree T of a digraph G is a tree, rooted at s (i.e., $ind_T(s) = 0$), covering all the vertices of K, and such that any pendant vertex u (i.e., $ind_T(u) = 1$ and $outd_T(u) = 0$) of T belongs to K. In addition, a K-tree whose s, K-diameter is at most D is called a D, K-tree.
- G is called a D, K-digraph, if every arc of G lies in some D, K-tree of G. We know that if G is a D, K-digraph, then node s has indegree 0; and every arc of G belongs to a dipath from s to some node in K; we say that G is rooted at s.

The following lemma gives a characterization of the minpaths M of $\mathcal{O}^D_{s,K}(E)$:

Lemma 5 *For a digraph $G = (V, E)$, terminal set K, vertex $s \in K$, and bound D, then M is a minpath of G if and only if M is a D, K-tree.*

From the definition of $R_{s,K}(G, D)$ and the previous definitions one gets

$$R_{s,K}(G, D) = \sum_{H \in \mathcal{O}^D_{s,K}(E)} \prod_{e \in H} p(e) \prod_{e \notin H} q(e). \quad (4)$$

As in the undirected case, the Diameter-constrained s, K-terminal reliability measure of a digraph G subsumes the classical Source-to-K-terminal reliability where no assumptions are made with regard to the lengths of the dipaths connecting the source vertex s to the terminal vertices of K. This equivalence allow us to conclude that in general to compute $R_{s,K}(G, D)$ is NP-hard, as the classical measure belongs to this computational class. In the next section we discuss the computational complexity of evaluating the Diamater-constrained s, K-terminal reliability.

3.1 Computational Complexity and DCR as a generalization of the classical reliability model

The classical *Source-to-K-terminal reliability* also known in the literature as *Reachability* of a digraph G, $R_{s,K}(G)$, is defined on the same probabilistic model (i.e., arcs fail randomly and independently with known probabilities and the vertices are always operational). The measure $R_{s,K}(G)$ is the probability that after the removal of the failing arcs there exists a dipath between the source-vertex s and v, for each $v \in K$. In this case there are not length restrictions of the dipaths joining s to the vertices of K, and by noting that the maximum length of a dipath joining a pair of vertices is of at most $n - 1$ edges, where n is the number of vertices of G, then

$$R_{s,K}(G) = R_{s,K}(G, n - 1). \quad (5)$$

This generalization of the classical reliability parameter allows us to reflect more stringent performance objectives by restricting the maximum length of a path in a network.

Let $G = (V, E)$ and K be a set of terminal vertices of G. For the classical reliability measure, computations of the s, K-terminal reliability for $K = V$ (known as the Source-to-all-terminal reliability) was shown to belong to the NP-hard computational class (see [20]). From these results and the fact that $R_{s,V}(G) = R_{s,V}(G, n-1)$, where n is the number of vertices of the graph G, and by restricting $D = n-1$, then

Theorem 6 *For the Diameter-constrained reliability, the computational complexity of computing $R_{s,V}(G, D)$ is NP-hard.*

Unlike the undirected case, at the present it is not known the computational complexity of evaluating $R_{s,K}(G)$ for arbitrary K or when $|K| = 2$, thus these problems also remain open for the diameter-constrained case.

In the next sections we discuss the definition of the domination invariant in the case of general coherent systems, and in the case of the diameter-constrained network reliability. The domination plays an important role in simplifying the calculation of the reliability for the directed case which it is nevertheless, as mentioned before, an intractable problem.

3.2 Domination

A graph invariant called the *reliability domination* of a graph G was introduced by Satyanarayana and Prabhakar [25] for the classical network reliability models, and has since been explored by several researchers in reliability theory [1, 2, 3, 14, 15]. The reliability domination plays an important role, allowing to efficiently implement the principle of Inclusion-Exclusion of probability theory applied to the evaluation of reliability measures for general reliability systems.

Let E be a finite set, and $P(E)$ be the power set of E. A nonempty subset $\mathcal{C} \subseteq P(E)$ is called a *clutter* of E if for any two elements $C_1, C_2 \in \mathcal{C}$, whenever $C_1 \subseteq C_2$, then $C_1 = C_2$. A pair $(E, \mathcal{C})$ will be referred to as a *system* and a system is *coherent* if each element of E is contained in some element of $\mathcal{C}$. A formation of $(E, \mathcal{C})$ is a collection of elements of $\mathcal{C}$ whose union yield E. The signed domination of the system $(E, \mathcal{C})$, denoted $d(E, \mathcal{C})$, is defined as the number of odd formations minus the number of even formations of E, where a formation is said to be odd or even if it is of odd or even cardinality respectively. Trivially by the previous definitions, a non-coherent system has no formations, so its signed domination is 0.

The clutters associated with the operation and failure of a specific element $x \in E$ are defined as follows. Let $\mathcal{C} - x = \{C - x : C \in \mathcal{C}\}$ and $\mathcal{C}_{-x} = \{C \in \mathcal{C} : x \notin C\}$. Now $\mathcal{C}_{-x}$ is clearly a clutter but $\mathcal{C} - x$ may not be one. We define $\mathcal{C}_{+x}$ to be the collection of elements of $\mathcal{C} - x$ which are not proper supersets of some element of $\mathcal{C} - x$. For an element $x \in E$, $\mathcal{C}_{-x}$ and $\mathcal{C}_{+x}$ are called the minors with respect to x of $\mathcal{C}$. Huseby [14, 15] showed the following result:

Theorem 7 *If $(E, \mathcal{C})$ is a system, with $x \in E$, and minors $\mathcal{C}_{-x}$ and $\mathcal{C}_{+x}$ of $\mathcal{C}$, then $d(E, \mathcal{C}) = d(E - \{x\}, \mathcal{C}_{+x}) - d(E - \{x\}, \mathcal{C}_{-x})$.*

We look now at the case of the diameter-constrained s, K terminal reliability of a digraph $G = (V, E)$ with $K \subseteq V$, $s \in K$, and diameter bound D. The system underlying our model is $(E, \mathcal{F}_{D,K}(G))$, where E is the set of arcs of G, and where $\mathcal{F}_{D,K}(G)$ is the collection of D, K-trees of G. A *formation* F of G is then a collection of D, K-trees of G whose union is E, the set of arcs of G. The *signed domination* of a digraph $G = (V, E)$, denoted $d(E, \mathcal{F}_{D,K}(G))$, with respect to a given subset $K \subseteq V$, $s \in K$, and bound D, is the number of odd minus the number of even formations of G.

For brevity, in what follows we will use the standard notation $\mathcal{C}$ to represent $\mathcal{F}_{D,K}(G)$, which is the clutter set in the diameter-constrained model. Also we denote the domination $d(E, \mathcal{F}_{D,K}(G))$ as $d_{D,K}(G)$. In addition, we observe that if x is an arc of G, then T is a D, K-tree of G such that $x \notin T$ iff T is a D, K-tree of $G - x$. Therefore $d(E - \{x\}, \mathcal{C}_{-x}) = d_{D,K}(G - x)$. Using this notation, the equation in Theorem 7 can be re-written as

$$\begin{aligned} d_{D,K}(G) &= d(E - \{x\}, \mathcal{C}_{+x}) - d(E - \{x\}, \mathcal{C}_{-x}) \\ &= d(E - \{x\}, \mathcal{C}_{+x}) - d_{D,K}(G - x) \quad (6) \end{aligned}$$

We next state the main results of this section, which are a characterization of the domination for diameter-constrained reliability models, and we discuss how these results can be used to compute the reliability of a network.

3.3 Characterization of the domination, application to reliability evaluation

Let $G = (V, E)$ be a digraph with terminal set K, $e = |E|$ arcs, $n = |V|$ vertices, and let D be the diameter bound. We define the following operation:

- $\mathcal{LP}(G, s, K)$. If G is s, K connected (i.e., there exists a dipath from s to any vertex $u \in K$ in G), this operation returns the length of the longest dipath from s to any vertex $u \in K$; otherwise it returns ∞.

The computation of $\mathcal{LP}(G, s, K)$ is in the NP-complete class; but it is of polynomial complexity if G is an acyclic digraph.

We observe that if G is not a D, K-digraph, there are some arcs in E which are not covered by any D, K-tree; so that the corresponding system is non-coherent, and there are no formations over the clutter $\mathcal{F}_{D,K}(G)$ able to cover E. As a result, the domination is zero. Consequently, from now on, we will restrict ourselves to the case of D, K-digraphs. For these digraphs, the domination is completely characterized by the following theorems (see [8]):

Theorem 8 *Let $G = (V, E)$ be a cyclic D, K-digraph with terminal set K, $n = |V|$ vertices, $n > 2$, and let D be the diameter bound. Then $d_{D,K}(G) = 0$.*

Theorem 9 *Let $G = (V, E)$ be a acyclic D, K-digraph with terminal set K, $e = |E|$ arcs, $n = |V|$ vertices, and let D be the diameter bound, then*

$$d_{D,K}(G) = \begin{cases} (-1)^{e-n+1} & : \mathcal{LP}(G, s, K) \leq D \\ 0 & : \textit{otherwise} \end{cases}$$

Both theorems can be proved by complete induction over the number of edges of the digraph, and using two different reductions, which preserve the existence of directed cycles and the relation of the length of the longest dipath and the diameter bound. One reduction is applied when all the nodes adjacent to s have indegree 1; in this case, we can "contract" all the arcs leaving s, and obtain a new digraph G^* with diameter bound $D - 1$ and with the same domination value as G. The other reduction is used when there is at least one node adjacent to s with indegree greater than 1; if $x = (s, u)$, then it can be proved that $d(E - \{x\}, \mathcal{C}_{+x}) = 0$, and by Equation 6 we have $d_{D,K}(G) = -d_{D,K}(G - x)$.

When $D = n-1$, we obtain the classical Source-to-K-terminal reliability model as a particular case. As all dipaths are of length smaller than n, then $\mathcal{LP}(G, s, K) \leq D$ unless the graph is not s, K-connected. Then the characterization reduces to the results in [24], i.e. that the domination is 0 if there is a directed cycle in G or G is not a D, K-digraph, and $(-1)^{e-n+1}$ otherwise.

These results are useful for computing the reliability of a given network. For a digraph $G = (V, E)$, terminal set K, and vertex $s \in K$, let $\mathcal{M} = \{M_1, M_2, \ldots, M_l\}$ be the set of minpaths of $\mathcal{O}^D_{s,K}(E)$. Define E_i to be the event that all the arcs of M_i operate. By Inclusion-Exclusion we obtain

$$R_{s,K}(G, D) = Pr\left(\bigcup_{i=1}^{l} E_i\right) = \sum_i Pr(E_i) - \sum_{i<j} Pr(E_i E_j) + \ldots + (-1)^{l+1} Pr(E_1 E_2 \ldots E_l), \quad (7)$$

where the event $E_i E_j \ldots E_m$ is the event that all the arcs of the subgraph obtained by the union of $M_i, M_j, \ldots, M_m$ are operating.

In Equation (7), the terms correspond to subgraphs obtained by the union of minpaths. As discussed previously, for the Diameter-constrained s, K-terminal reliability of a digraph G, with terminal set K, vertex $s \in K$, and diameter bound D, the minpaths are D, K-trees, the formations are sets of minpaths, and the subgraphs are D, K-digraphs. The same D, K-digraph can be obtained from different formations; this means that it may appear more than once, sometimes with positive sign, and sometimes with negative sign, depending if the corresponding formation has an odd or or an even number of D, K-trees. In fact, its net contribution will be exactly the number of odd minus the number of even formations of the graph, i.e., its domination invariant. Thus using these facts and the above definitions, we can rewrite Equation (7) as

$$R_{s,K}(G, D) = \sum_{H \in \mathcal{H}} d_{D,K}(H) Pr(H), \quad (8)$$

where $\mathcal{H}$ is the class of all D, K-digraphs of G, and $Pr(H)$ is the probability that the arcs of H are operative.

3.4 Algorithm

In this section we present an algorithm for the computation of the Diameter-constrained s, K-terminal reliability based upon Equation 8 and the characterization of the domination stated in Theorem 8 and Theorem 9 [8].

It is easy to see that a digraph with parallel arcs $\{e_1, e_2, \ldots, e_m\}$ emanating from a node u, and directed into a node v, and with corresponding reliabilities $\{p(e_1), p(e_2), \ldots, p(e_m)\}$, can be replaced by a single arc $e = (u, v)$ with reliability

$$p(e) = 1 - \prod_{i=1}^{m} (1 - p(e_i)), \quad (9)$$

without affecting the reliability; thus we are only concerned with digraphs without parallel arcs.

For a digraph $G = (V, E)$, with terminal set $K \subseteq V$, and distinguished vertex $s \in K$, we say that G is s, K-*connected* if there exists in G a s, u-dipath for every $u \in K$. If $ind_G(s) = 0$, we will denominate this graph s-rooted, and from this point on we will be only concerned with s-rooted digraphs, since if that is not the case, then $d_{D,K}(G) = 0$, as stated in the following claim:

Claim 10 *Suppose that $G = (V, E)$ is a digraph with terminal set $K \subseteq V$, and vertex $s \in K$. If $ind_G(s) > 0$ then $d_{D,K}(G) = 0$.*

We next need to define irrelevant arcs:

Definition 11 *Given a digraph $G = (V, E)$, with terminal set $K \subseteq V$, vertex $s \in K$, an arc $e = (u, v) \in E$ is an irrelevant arc if at least one of the following is true:*

(a) The arc e belongs to a connected component $G' = (V', E')$ of G, where $V' \subseteq V - K$.

(b) The vertex $u \in V - K$ has $ind_G(u) = 0$.

(c) The vertex $v \in V - K$ has $outd_G(v) = 0$.

According to Theorem 8 and Theorem 9, the algorithm should only be concerned in identifying acyclic D, K-digraphs whose longest s, u-dipath, $u \in K$, is of length at most D. The following Lemma gives a sufficient condition for such digraphs.

Lemma 12 *Given a digraph $G = (V, E)$, with terminal set K, and vertex $s \in K$, suppose that G is an acyclic, s, K-connected digraph, with no irrelevant arcs, and $\mathcal{LP}(G, s, K) \leq D$, then G is a D, K-digraph.*

We now an algorithm for efficiently generating precisely all these digraphs having non-null domination.

As a first step, we assume that G is s-rooted. If this is not the case we can simply delete any arc directed into s, obtaining a s-rooted digraph. Moreover parallel arcs are replaced by a single arc with reliability obtained as explained at the beginning of the section.

The algorithm has five stages.

(a) Determine of G has irrelevant arcs. If that is the case, generate a digraph from G by deleting these arcs, and any isolated vertex $u \in V - K$ obtained from this deletion.

(b) Determine if G is s, K-connected. If G is not s, K-connected, then we do not generate any subgraphs from G.

(c) If G contains a dicycle, generate acyclic subgraphs of G.

(d) If G is acyclic, determine if $\mathcal{LP}(G, s, K) > D$. If that is the case, generate all possible acyclic subgraphs G' of G such that $\mathcal{LP}(G', s, K) \leq D$.

(e) If G is acyclic and $\mathcal{LP}(G, s, K) \leq D$, then generate all possible subgraphs of G.

Generation of duplicate subgraphs at all stages is completely avoided by a simple check.
The algorithm grows a rooted directed tree with the following properties:

1. Vertices represent nonempty subgraphs of G, the root vertex being G itself. Any vertex, say r, corresponds one-to-one with the subgraph G_r which is of one of the following five types: a) G_r contains irrelevant arcs, b) G_r is not s, K-connected, c) G_r is s, K-connected and cyclic, d) G_r is s, K-connected, acyclic, and $\mathcal{LP}(G, s, K) > D$, e) G_r is s, K-connected, acyclic, and $\mathcal{LP}(G, s, K) \leq D$.

2. A link directed from vertex i to vertex j of the tree is labeled X, where X represents the set of arcs deleted from G_i to obtain G_j.

Additional Definitions (directed tree generation):
Father (Child): Vertex $i(j)$ is the father (child) of $j(i)$ when there exists an link directed from i to j.

Ancestor: Vertex i is the ancestor to j when i is contained in the path from the root vertex to j ($i \neq j$).

Brother: Vertices having the same father are termed brothers.

Younger (Elder) Brother: A vertex i is the younger (elder) brother of vertex j, if the algorithm generates the children of vertex i later (earlier) than the children of vertex j.

Rooted Directed Tree Generation:
Starting from the root vertex, the algorithm grows the tree progressively generating children, if any, of every vertex. There are five rules for generating the children of vertex r, depending on the nature of G_r.

Rule 1 *G_r has irrelevant arcs*. Let X' be the label corresponding to the set of irrelevant arcs of G_r. In this case generate a new node representing the digraph obtained from G_r by deleting these arcs (and possibly any isolated vertices obtained from this deletion), provided $X' \cap X = \emptyset$, where X is the label of the link incident into the elder brothers of r or elder brothers of an ancestor of r; otherwise do not generate any children from G_r.

Rule 2 *G_r is not s, K-connected*. In this case G_r does not generate any children.

Rule 3 *G_r is s, K-connected and cyclic*. Consider a dicycle C in G_r containing the arcs $e_1, e_2, \ldots, e_c$. Then $G_{r_j} = G_r - e_j$, $(j = 1, 2, \ldots, c)$, is a child of G_r, provided $\{e_j\} \cap X = \emptyset$, where X is the label of a link incident into the elder brothers of r or elder brothers of an ancestor of r. Determination of a dicycle is determined by application of Depth First Search (applied in Rule 2). Clearly a state $G_r - e_j$ where e_j does not belong to the dicycle C, contains also C, thus by Theorem 8, $d_{D,K}(G_k - e_j) = 0$, so it is not necessary to generate this state.

Rule 4 *G_r is s, K-connected, acyclic, and $\mathcal{LP}(G_r, s, K) > D$*. Consider a longest s, u-dipath L in G_r containing the arcs $e_1, e_2, \ldots, e_l$. Then $G_{r_j} = G_r - e_j$, $(j = 1, 2, \ldots, l)$, is a child of G_r, provided $\{e_j\} \cap X = \emptyset$, where X is the label of a link incident into any elder brother of r or elder brother of an ancestor of r. Determination of a longest s, u-dipath is determined by application of a longest path algorithm (as used in CPM and PERT applications for example, see [17]) which can execute in time complexity $O(|V| + |E|)$ for acyclic digraphs. It is not necessary to consider a state $G_r - e_j$ where e_j does not belong to the dipath L, because $G_r - e_j$ is either not s, K-connected and its domination is 0, or it is s, K-connected and contains the path L of length greater than D, and by Theorem 9 its domination is also 0.

Rule 5 G_r *is* s, K*-connected, acyclic, and* $\mathcal{LP}(G_r, s, K) \leq D$. Let $G_r = (V_k, E_k)$. Assuming that G_r does not have irrelevant arcs, it follows from Lemma 12 that G_r is a D, K-digraph, therefore contributing to the total reliability by $(-1)^{|E_r|-|V_r|+1} \prod_{e \in E_k} p(e)$. Moreover let $G_{r_j} = G_r - e_j$, $e_j \in E_r$ be a child of G_r, provided $\{e_j\} \cap X = \emptyset$, where X is the label of a link incident into any elder brother of r or elder brother of an ancestor of r.

Of the possible $2^{|E|}$ states (i.e., digraphs) to be evaluated, rules $1, 3,$ and 4 of the above algorithm represent a significant reduction on the total number of executable operations performed, since many states are avoided, especially when the digraphs contain irrelevant arcs, they contain several directed cycles, or the diameter bound D is small.

4 Conclusions

In this paper we presented a summary of combinatorial as well as computational properties of the Diameter-constrained network reliability where we extended well-known properties of the classical network reliability measure.

As the computational complexity of evaluating the DCR is NP-hard, several problems are still open, as it is for example application of meta-heuristics (e.g., Monte Carlo techniques) to obtain an approximation of this measure and determination of classes of graphs (e.g., series-parallel graphs) for which polynomial time algorithms for determination of the reliability exist.

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