Efficient evaluation of a Diameter-Constrained reliability measure of some families of graphs

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Abstract

Given an undirected graph $G = (V, E)$, two distinguished vertices s and t of G, and a diameter bound D, a D-s,t-path is a path between s and t composed of at most D edges. An edge e or vertex u are called D-irrelevant if e or u does not belong to any D-s,t-path of G. In this paper we study the problem of efficiently detecting D-irrelevant edges and vertices and also study the computational complexity of diameter-related problems in graphs. Detection and subsequent deletion of D-irrelevant edges and vertices have been shown to be fundamental in reducing the computational effort to evaluate the Source-to-terminal Diameter-Constrained reliability of a graph G, $R_{\{s,t\}}(G, D)$, which is defined as the probability that at least a path between s and t, with at most D edges, survives after deletion of the failed edges (under the assumption that edges fail independently and nodes are perfectly reliable). Even though the computation of $R_{\{s,t\}}(G, D)$ has be shown to be NP-Hard for fixed values of the diameter bound D (i.e., D is independent of the size of a given graph), we show that the calculation of this reliability measure can be determined in time $2^{o(D^2)}$ for some families of graphs, yielding an efficient determination of the reliability for small values of D.
KEYWORDS: Diameter, euclidean distance, reliability, optimization.

1 Introduction

Unless otherwise stated, in this paper we consider undirected graphs $G = (V, E)$, where V represents a finite set of vertices and E is a finite set of edges.

In this paper we investigate, from a computational point of view, diameter properties of graphs related to the following optimization problem: given two vertices s and t of G, we would like to efficiently identify edges and/or vertices that do not belong to any path between s and t of length less or equal to a given bound D; we then apply some of the results shown to compute more efficiently the Diameter-Constrained reliability (DCR) of a communication network (originally introduced in [18]), a constrained version of the classical network reliability measure (refer to [21, 22, 23, 25, 26] for further discussion on this classical model). As the computation of the *DCR* is known to be NP-Hard, even when the diameter bound D is a constant value independent of the size of a graph, we show there exist families of graphs in which its computation can be achieved in time $2^{o(D^2)}$.

Given a probabilistic graph $G = (V, E)$, a set of terminal nodes $K \subseteq V$, and a diameter bound D, in which each edge $e \in E$ has been assigned a probability of failure $q(e) = 1 - r(e)$ ($r(e)$ is called the reliability of the edge e) under the assumption that edges fail independently and nodes are perfectly reliable, the *K-terminal Diameter-Constrained* reliability, $R_K(G, D)$, gives the probability of the event that for each pair of nodes $x, y \in K$, a path between x and y of length (i.e., number of edges comprising the path) D or less, called a D-x, y-path, survives after deletion of the failed edges. In this paper we consider the case when $K = \{s, t\}$, known as the Source-to-terminal Diameter-Constrained reliability of a graph G, denoted as $R_{\{s,t\}}(G, D)$. For the classical reliability measure, the K-terminal reliability $R_K(G)$, of a graph G, is the probability that after the removal of the failed edges, each pair of nodes $x, y \in K$ is connected by at least an operational path, independently of its length. Both the classical reliability and the *DCR* can be computed by application of a theorem of Moskowitz [14], also refered as the *Factoring Theorem*, in which the reliability of the probabilistic graph is computed in term of the reliabilities of two smaller graphs using an specific edge as pivot.

In many real-life situations the quality of the communication depends on the existence of a path connecting each pair of terminals x and y, whose length (measured as the number of edges) is bounded by a given integer D. The K-terminal Diameter-Constrained Network reliability is the probability of this event and it was originally introduced by Petingi and Rodriguez in 2001 [18] (for a survey on this reliability model refer to [16]). The *DCR* can be applied to assess performance objectives of for example packet-oriented networks where links may fail and there is a "time-to-live" (TTL) limit, specified in number of hops that can be traversed by any given packet (for instance IPv6 packets include a hop limit field [11]). It is also the case of many overlay networks (such as peer-to-peer file sharing networks) that employ flooding protocols for peer discovery which specify a maximum number of hops to be visited by a request (for instance Gnutella, which employs a flooding-based routing algorithm with a TTL value of 7 hops [20]). As the classical reliability measure does not capture the distances between the network nodes, the *DCR* can be applied to assess the probability of establishing a connection by setting, for example, the diameter bound D equal to the maximum number of hops to be visited by a packet or request. Another scenario for P2P networks is obtained if the link reliability value represents the probability that a pair of given nodes are in each other routing tables. In that case, the *DCR* models the fraction of the peers than can be reached from an arbitrary node.

The *DCR* measure subsumes the classical reliability in the following sense; as any path in a network on n vertices is composed of at most $n-1$ edges, then

$R_K(G, D) = R_K(G)$, whenever $D = n-1$. As calculation the classical reliability for arbitrary terminal set K is an NP-Hard problem [1], then evaluation of the *DCR* is an NP-Hard problem as well. For fixed number of terminal vertices K, and for fixed values of the bound D, Cancela and Petingi [7] proved that to determine $R_K(G, D)$ is also an NP-Hard problem. Monte Carlo techniques have been shown to be excellent candidates to accurately estimate the classical reliability [4, 5] as well as to calculate the *DCR* [6].

For the classical reliability, topological reductions (e.g., series reduction, degree-2 reduction, parallel reduction, polygon-to-chain reductions) can be applied to a graph to reduce its size while preserving the reliability, and therefore reducing the computational effort for its evaluation. A series-parallel graph is a graph $G = (V, E)$ that can be reduced to single edge by application of series-parallel reductions, and, for any arbitrary set of terminal vertices K, $R_K(G)$ can be computed in time $O(|E|)$ [22]. For fixed values of the diameter bound D, with the exception of parallel reductions, the aforementioned transformations cannot be applied to evaluate the *DCR*, as these topological transformations may change the distance between nodes of a graph (and therefore its diameter). As for this case these transformations are not reliability-preserving, the deletion of D-irrelevant edges and vertices can be considered as the alternative to efficiently calculate $R_K(G, D)$, as well as possibly determine families of graphs in which $R_K(G, D)$ can be calculated in polynomial time.

An edge e or vertex u of a graph G are said to be irrelevant if deletion of the edge e (denoted as $G-e$) or deletion of the vertex u (denoted as G/u) preserves the K-terminal Diameter-Constrained reliability, that is, $R_K(G, D) = R_K(G - e, D)$, or $R_K(G, D) = R_K(G/u, D)$, for a given diameter bound D, respectively. For the specific case when $K = \{s, t\}$, an edge e or vertex u that do not not belong to any D-s, t-path can be then deleted without affecting the reliability. Preliminary study addressing D-irrelevancy with respect to edges was originally presented [8], and recently in [17] to show how the computational effort to evaluate the Source-to-terminal Diameter-Constrained reliability of graphs can be improved when irrelevant edges are efficiently detected (and then deleted); among other results we are extending this study by presenting new sufficient conditions to identify D-irrelevant vertices.

The paper is structured as follows. In Section 2 we introduce notation and definitions that will be used in the sequel. In Section 3 we study the following optimization problem: given two vertices s and t of G, we would like to efficiently identify edges and/or vertices that do not belong to any path between s and t of length less or equal to a given bound D; we then introduce new sufficient conditions, in addition to the ones presented in [8], and in [17], to efficiently identify irrelevant vertices in graphs. In Section 4, we show how these sufficient conditions can be embedded within the frame of a procedure based on Moskowitz's Theorem (i.e., Factoring Theorem) to evaluate $R_{\{s,t\}}(G, D)$. In Section 5 we show that for the family of *Grids* (planar graphs where each internal phase is a square), and for fixed values of D, $R_{\{s,t\}}(G, D)$ can be calculated in time $2^{o(D^2)}$. Finally, in Section 6, we present conclusions and further research.

2 Definitions and notation

In this section we introduce definitions and notation that will be used in the sequel.

Definition 1 *(Path) - An s,t-path is a sequence of distinct vertices $< u_0 = s, u_1, u_2, \ldots, u_{r-1}, t = u_r >$ of a graph G, such as (u_i, u_{i+1}) is an edge of G, $0 \leq i \leq r-1$. If each edge $e = (u, v)$ is assigned an integer weight $w(e)$, the length of an s,t-path $p =< u_0 = s, u_1, u_2, \ldots, u_{r-1}, t = u_r >$ is $L_G(p) = \sum_{i=0}^{r-1} w((u_i, u_{i+1}))$. For the unweighted case, or equivalently when each edge is assigned a weight of one, then the length $L_G(p)$ is the number of edges comprising the path p.*

Definition 2 *(Simple Cycle) - A simple cycle is a sequence of vertices $< u_0 = x, u_1, u_2, \ldots, u_{r-1}, y = u_r >$ of a graph G, such as (u_i, u_{i+1}) is an edge of G, $0 \leq i \leq r-1$, and all the vertices of the sequence are distinct except for $x = y$.*

Definition 3 *(Irrelevancy and Critical graphs) - An s,t-path p is called a D-s,t-path if $L(p) \leq D$ (i.e., the path length is at most D). Given a graph $G = (V, E)$, a distinguished set of terminal vertices set $K = \{s, t\}$, and a diameter bound D, and edge $e = (u, v) \in E$ is said to be D-relevant if e lies in some D-s,t-path, otherwise e is D-irrelevant. If every edge of G is D-relevant then G is called D-diameter-critical.*

Definition 4 *(Distance and Diameter) - The distance between two vertices x, y of G is $distance_G(x, y) = min\{L_G(p) : p \text{ is an } x,y\text{-path of } G\}$. Moreover the K-diameter is the maximum distance between vertices of K.*

Definition 5 *(Degree) - The degree of a vertex v of a graph G, denoted as $deg_G(v)$, is the number of edges incident at v.*

We are also introducing notation to describe topological transformations in graphs.

- Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and two distinct vertices a, b such that $a \in V_1$ and $b \in V_2$, let $G_1 a.e.b G_2$ describe the operation of joining G_1 and G_2 by an edge $e = (a, b)$.
- Given two paths $p_1 =< u_0, u_1, u_2, \ldots, u_{r-1}, u_r >$, and $p_2 =< v_0, v_1, v_2, \ldots, v_{m-1}, v_m >$, and two distinct vertices u and v of p_1 and p_2, respectively, let $p_1 u.e.v p_2$ describe the operation of joining p_1 and p_2 by an edge $e = (u, v)$.

In the following section we study the problem of D-irrelevancy to efficiently identify irrelevant edges and/or vertices for fixed values of the diameter bound D.

3 Reliability preserving transformations for Diameter Constrained reliability measure

Given an edge e and two vertices s and t of a graph G, e is D-relevant if and only if there exist an s,t-path p among all s,t-paths containing e, such as

$L_G(p) \leq D$. Thus we are considering first the optimization problem of finding a shortest s,t-path of G containing an specific edge e.

For the case in which negative integer weights can be assigned to the edges of the graph, we show next that to find a shortest s,t-path containing an specific edge e is NP-Hard, by transforming the *Longest Path* problem in which positive integer weights are assigned to the edges of a graph, into this decision problem:

P1: *Shortest Path Containing an Specific Edge (SPE)*

Instance: Graph $G' = (V', E')$, edge e^*, vertices $s', t' \in V'$, length $l(e') \in \mathcal{Z}$, for each $e' \in E'$, and bound $k' \in \mathcal{Z}$.

Question: Is there an s', t'-path p' of length k' or less containing the edge e^* (i.e., $L(p') \leq k'$)?

Consider next the Longest Path problem, to be known to belong to the NP-Complete computational class, when positive integer weights are assigned to the edges [13]:

P2: *Longest Path (SPP)*

Instance: Graph $G = (V, E)$, vertices $s, t \in V$, length $l(e) \in \mathcal{Z}^+$, for each $e \in E$, and bound $k \in \mathcal{Z}^+$.

Question: Is there an s,t-path p of length k or greater (i.e., $L(p) \geq k$)?

Lemma 1 *The Shortest Path Containing an Specific Edge problem,* **P1***, is NP-Hard.*

Sketch of Proof: It is not known if SPE is in NP. Consider the transformation from the Longest Path problem to the *SPE* problem.

Let $G = (V, E)$, vertices $s, t \in V$, length $l(e) \in \mathcal{Z}^+$, bound $k \in \mathcal{Z}^+$ be an instance of **P2**. Let $G' = (V', E')$ be graph obtained by joining two copies of G by an edge $e^* = (a, b)$ where a is vertex t of the first copy of G and b is vertex s of the second copy of G, that is, $G' = Gt.e^*.sG$. Moreover let the vertices s' and t' be the vertices of G' corresponding to vertex s of first copy of G and vertex t of the second copy of G, respectively. In addition, for each edge $e' \in E'$, let $l(e') = -l(e)$ where e is e' corresponding edge in G (as the edges of G have been assigned positive weights, then the edges of G' have negative weights). Moreover let $l(e^*) = -1$, and $k' = -2k - 1$ (see Figure 1).

It is not difficult to show then that G contains an s,t-path p of length $L_G(p) \geq k$ if and only if G' contains and s', t'-path p' of length $L_{G'}(p') \leq -2k - 1 = k'$. ∎

If the optimization problem *SPE* is extended to find a shortest s,t-path containing a predefined set of edges (or vertices), then the problem is NP-Hard as well, as the above proof can be generalized by replacing the edge e^* by a path with a fixed number of edges and vertices, where each edge of the path is assigned a weight of -1.

When positive weights are assigned to the edges, little is known regarding the corresponding optimization problem of finding a shortest s,t-path containing

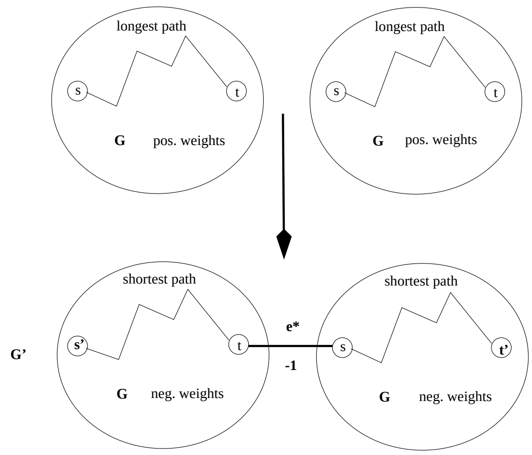

Figure 1: Transformation from the Longest Path problem to the Shortest Path Containing an Specific Edge

an specific edge (or a predefined set of edges), even though this problem has been studied since the 1960s. In a recent paper by Bijörklund et al. [2], the authors studied the problem of finding a shortest simple cycle through a set of elements $B \subset V \cup E$ (i.e., vertices and edges), and a randomized algorithm (of order $2^{|B|}n^{O(1)}$) was introduced for any arbitrary set B of elements. This problem is equivalent to the problem of finding a shortest s,t-path through a set of edges (or vertices), as an edge between s and t can be added, if s and t are not adjacent, to form a simple cycle. However, according to this paper, little is known regarding the optimal solution of this optimization problem when $b = |B|$ is a constant, $b \geq 3$. Thus it is not clear if there exist a computationally efficient way to determine necessary and sufficient conditions to classify an edge e and/or vertex u as D-irrelevant or not. However in [8], sufficient conditions to identify D-irrelevant edges were introduced, and are stated in the following proposition:

Proposition 1 *Given a graph $G = (V, E)$, a diameter bound $D \in \mathcal{Z}^+$, and an edge $e = (u, v)$ of G. If $distance_G(s, u) + distance_G(v, t) \geq D$ and $distance_G(s, v) + distance_G(u, t) \geq D$, then e is D-irrelevant.*

Given a graph $G = (V, E)$, let $n = |V|$ and $m = |E|$. The distance (see Definition 4) between s and any other vertex of the graph, can be efficiently determined by application of Dijkstra's Shortest Path algorithm (**DSPA** for short) [12] of order $O(m + n)$. From Definition 3, it follows that edges that do not belong to any s,t-path are also irrelevant; it is important to note that if G is one-connected (i.e., there exist a vertex, called a *cut-point*, whose deletion disconnects G), and s and t belong to a two-connected component $C = (V_c, E_c)$, any edge e that does not belong to this component, is also D-irrelevant as no

s, t-path containing e exists in G; thus is possible that Proposition 1 won't recognize e as irrelevant. In this case, edges that belong to blocks (two-connected components) other than C, can be efficiently identified by a linear-time algorithm based on bi-connectivity theory [10]. Next suppose that G has more than one connected component, then if s and t belong to a same connected component (not necessarily a two-connected one) $C = (V_c, E_c)$, any edge e that does not belong to C is also irrelevant. In this case Proposition 1 will classify this edge as D-irrelevant as the original distance between s and a end-point of e is set to infinite by Dijkstra's algorithm.

In [17] additional sufficient conditions to detect a superset of the set of D-irrelevant edges identified by Proposition 1, were introduced:

Proposition 2 *Given a graph $G = (V, E)$, a diameter bound $D \in \mathcal{Z}^+$, and an edge $e = (u, v)$ of G. If $distance_{G-e}(s, u) + distance_{G-e}(v, t) \geq D$ and $distance_{G-e}(s, v) + distance_{G-e}(u, t) \geq D$, then e is D-irrelevant.*

It is obvious that if the conditions stated in Proposition 1 recognize a set of D-irrelevant edges S_1, then the conditions specified by Proposition 2 find a set of D-irrelevant edges S_2 where $S_1 \subseteq S_2$.

A procedures **Irrelevant-P_1** to identify (and delete) irrelevant edges based on Proposition 1 was presented in ([8]) and a procedure **Irrelevant-P_2** also to detect (and delete) irrelevant edges based on Proposition 2 was presented in [17].

Despite of the fact that procedure **Irrelevant-P_2()** finds a superset of the edges detected by **Irrelevant-P_1()**, the conditions stated in Step 2 of the latest are determined by just two application of Dijkstra's algorithm in time $O(m+n)$, while the distance conditions stated in Proposition 2 are determined in time $O(m^2)$, as we must apply **DSPA** m times, each time when we delete a possible irrelevant edge e from G; the trade-off between the number of irrelevant edges identified, and the computational complexity for detecting these edges, when applying these procedures, was further investigated in [17].

Next consider the following proposition to find D-irrelevant vertices:

Proposition 3 *Given a graph $G = (V, E)$, a diameter bound $D \in \mathcal{Z}^+$, and a vertex u of G. If $distance_G(s, u) + distance_G(u, t) > D$ then u is D-irrelevant.*

Proof: Suppose that $distance_G(s, u) + distance_G(u, t) > D$ and G contains an s, t-path $p =< u_0 = s, u_1, u_2, \ldots, u, \ldots, u_{r-1}, u_r = t >$, and consider the subpaths $p_1 =< u_0 = s, u_1, u_2, \ldots, u >$ and $p_2 =< u, \ldots, u_{r-1}, u_r = t >$ of lengths l_1 and l_2, respectively. But the length of l_1 is at least $distance_G(s, u)$ and the length of l_2 is at least $distance_G(u, t)$, thus the length of p is at least $D + 1$. ■

A procedure **Irrelevant-P_3** to identify D-irrelevant vertices based on Proposition 3 can be implemented by application Dijkstra's Shortest Path algorithm in time $O(m + n)$.

4 Moskowitz and computation of the Source-to-terminal Diameter-Constrained Reliability

Moskowitz's Decomposition Theorem express the reliability of a network G as a function of the reliabilities of the two networks obtained from G by fixing the state of a selected edge e either up (i.e., r_e is set to 1) or down (i.e., r_e is set to 0). Moskowitz's decomposition was extensively used within the context of the classical reliability (see [15, 19, 23, 25, 26]). Within this context we say that the random state of an edge e is *undetermined* if $0 < r_e < 1$ [8].

Theorem 1 *For any network G that has at least one edge e whose random state is undetermined then*

$$\begin{aligned} & R_{\{s,t\}}(G, D) = \\ & \quad r_e R_{\{s,t\}}(G * e, D) + \\ & \quad (1 - r_e) R_{\{s,t\}}(G - e, D), \ \textit{where} \end{aligned}$$

- *e is an edge with undetermined random state in G if $0 < r_e < 1$.*
- *$G * e$ is the network obtained from G by fixing the edge e up (i.e., $r_e = 1$).*
- *$G - e$ is the network obtained from G by fixing edge e down (i.e., $r_e = 0$, or equivalently e is deleted from G).*

Consider a procedure **Factoring()** to evaluate $R_{\{s,t\}}(G, D)$, derived from Theorem 1; this procedure describes a binary tree in which each node j of this tree represents a subgraph of G, G_j, in which its edges are either operational (reliability 1), have failed (reliability 0), or whose random states are undetermined (the root node of the derived binary tree represents the original network G). For each of the possible subgraphs G_j's, its Source-to-terminal Diameter-Constrained reliability is then calculated as:

$$R_{\{s,t\}}(G_j, D) = \begin{cases} 0: \text{ if there is no } D\text{-}s,t\text{-path in } G_j. \\ 1: \text{if } G_j \text{ contains an operational } D\text{-}s,t\text{-path.} \\ r_e R_{\{s,t\}}(G_j * e, D) + (1 - r_e) R_{\{s,t\}}(G_j - e, D): \ e \text{ is undetermined.} \end{cases}$$

We can embed procedure **Irrelevant-P_3()**, described in Section 3, within **Factoring()**, to possibly delete irrelevant vertices (and their corresponding edges incident at them) in each of the states G_j of the binary tree generated by the application of the recursive function previously stated, to possibly shorten the computational effort. The following procedure, called **Fact-Reductions-3()**, evaluates the Source-to-terminal Diameter-Constrained reliability of a network while enforcing the deletion of irrelevant vertices (and corresponding edges incident at them) by procedure **Irrelevant-P_3**. Procedure **Fact-Reductions**$-3()$ receives five parameters, namely the network topology G, the source and terminal nodes s and t, the diameter constraint D, and a flag called $flagr$, which indicates whether further reductions are or are not possible. At the first invocation, $flagr$ is set to 1.

Procedure **Fact-Reduction**$-3(G, s, t, D, flagr)$ Input: network $G = (V, E)$, s, t, D, and $flagr$
Output: reliability $R_{\{s,t\}}(G, D)$

1. Check end recursion condition:
   - 1.1. If G contains a D-s, t-path having only operational edges return (1).
   - 1.2. If there is no D-s, t-path in G return (0).
2. Apply procedure to detect irrelevant vertices:
   - 2.1. If $(flagr = 1)$ *call* **Irrelevant-P_3**(G, s, t, D).
3. Select randomly an edge e in G with undetermined state.
4. Solve recursively for $G - e$: $R_{\{s,t\}}(G - e, D) =$ **Fact-Reduction**$-3(G - e, s, t, 1)$.
5. Solve recursively for $G * e$: $R_{\{s,t\}}(G * e, D) =$ **Fact-Reduction**$-3(G * e, s, t, 0)$.
6. Compute $R_{\{s,t\}}(G, D)$: return $(R_{\{s,t\}}(G, D) = (1 - r_e)R_{\{s,t\}}(G - e, D) + r_e R_{\{s,t\}}(G * e, D))$.

In the next section we show a family of graphs called *Grids*.

5 Families of graphs in which the Source-to-terminal Diameter Constrained reliability can be computed in polynomial time

The *DCR* measure subsumes the classical reliability, that is, $R_K(G, D) = R_K(G)$, whenever $D = |V| - 1$. As calculation the classical reliability for arbitrary terminal set K was shown to be an NP-Hard problem [1], then evaluation of the *DCR* is an NP-Hard problem as well. For fixed number of terminal vertices K, and for fixed values of the bound D, Cancela and Petingi [7] proved that to determine $R_K(G, D)$ is also an NP-Hard problem.

A *Grid* is a planar graph in which each phase is a square. In this Section we show that for fixed values of the diameter bound D, if $G = (V, E)$ is a Grid, then $R_{\{s,t\}}(G, D)$ can be calculated in time $2^{o(D)}$.

For the next proof we embed a *Grid* graph $G = (V, E)$ in a plane in which each square is a unit of area, and whose origin is the midpoint determined by the coordinates of s and t (see Figure 2).

Lemma 2 *Let $G = (V, E)$ be a* Grid *graph, with distinguished set of vertices s and t, and D be the given diameter bound. Morever let $distance_G(s,t) \leq D$ (otherwise $R_{\{s,t\}}(G,D) = 0$). Embed the* Grid *graph in a plane whose origin is the midpoint $M_{s,t}$ between s and t, and consider the circle $C_{\{s,t\}}(G,D)$ circumscribed by a radius of length $D/2$ as shown in Figure 2. Then any vertex outside $C_{\{s,t\}}(G,D)$ is D-irrelevant.*

Sketch of Proof: Let s and t have coordinates (x_s, y_s) and (x_t, y_t), respectively. In addition let $b = (x_c, y_c)$ be a point of the circumference at distance $D/2$ from the origin. Suppose that r_1 and r_2 are the distances from s and t to $b = (x_c, y_c)$, respectively. From analytical geometry it follows that $r_1 + r_2 \geq 2.D/2 = D$. As $r_1 \leq distance_G(s,b)$, and $r_2 \leq distance_G(t,b)$, then for any vertex d outside $C_{\{s,t\}}(G,D)$, then $distance_G(s,d) + distance_G(t,d) > D$ therefore d is D-irrelevant according to Proposition 3. ■

An upper bound on the maximum number of faces contained in the circle $C_{\{s,t\}}(G,D)$ can be easily determined. As the area of a square is a unit, then the number of faces (i.e., squares) is the area of $C_{\{s,t\}}(G,D)$.

Lemma 3 *The circle $C_{\{s,t\}}(G,D)$ contains at most $\lceil \pi.(D/2)^2 \rceil = \lceil \frac{\pi}{4} D^2 \rceil$ faces.*

In the following lemma we determine the maximum number of edges contained in $C_{\{s,t\}}(G,D)$

Lemma 4 *The circle $C_{\{s,t\}}(G,D)$ contains at most $4\lceil \pi.(D/2)^2 \rceil - 1 = 4\lceil \frac{\pi}{4} D^2 \rceil - 1$ edges.*

Proof: We want to first determine what is the maximum number of vertices contained in a connected *Grid* graph. The proof is by construction. First consider a *Grid* composed of one square (one face). Suppose that we want to add another face while maximizing the number of vertices, and clearly the optimal arrangement is obtained when the two squares are joined by a single vertex, thus incrementing the number of faces by 1 and the number of vertices by three. Any other arrangement will produce less number of vertices while increasing the number of faces by one. Thus we obtain the set of ordered pairs $\{(1,4), (2,7), (3,10), ...(f, 3f+1)\}$ where the first integer of each pair is the number of faces and second is the number of vertices.

Consider the *Grid* circumscribed by $C_{\{s,t\}}(G,D)$, and call it $G' = (V', E')$. Let f', n', and m' represent the number of faces, vertices, and edges of G', respectively. Then as $f' + n' = m' + 2$ and n' is at most $3f' + 1$, then $m' \leq 4f' - 1$. Thus from Lemma 3 one gets $m' \leq 4\lceil \frac{\pi}{4} D^2 \rceil - 1$. ■

Corollary 1 *For any* Grid *graph, and for fixed value of the diameter bound D, calculation of $R_{\{s,t\}}(G,D)$ takes at most $2^{4\lceil \frac{\pi}{4} D^2 \rceil - 1}$ steps.*

Proof: Consider Procedure **Fact-Reduction**$-3()$ described in Section 4. We will first delete all D irrelevant vertices by application of Procedure **Irrelevant-**$P_3()$ based on Proposition 3 (see Section 3). Deletion of these vertices will be accomplished by application of Dijkstra's Shortest Path algorithm. After this

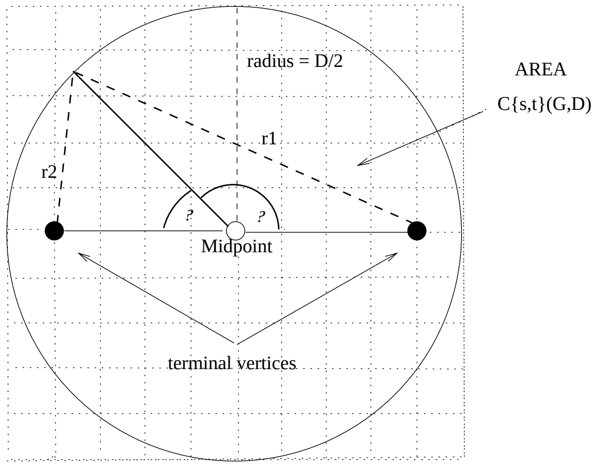

Figure 2: Embedding a *Grid* graph in a plane

deletion step, and according to Lemma 4, a graph $G' = (V', E')$ with at most $4\lceil \frac{\pi}{4} D^2 \rceil - 1$ edges is then obtained from G. Assuming that no other deletion is performed (which will remain an open problem), as **Fact-Reduction**$-3()$ describes a binary tree on the number of edges, then calculation of $R_{\{s,t\}}(G, D)$ takes at most $2^{4\lceil \frac{\pi}{4} D^2 \rceil - 1}$ steps. ∎

The significance of Corollary 1 is that for small values of the diameter bound D, calculation of the Source-to terminal Diameter Constrained reliability can be determined efficiently, independently of the size of *Grid* graph under consideration.

6 Conclusions and further research

The purpose of this paper was to introduce new reliability-preserving sufficient conditions to identify D-irrelevant vertices. By application of these conditions, and by embeding a graph in a plane, we showed that independently of the size of any *Grid* graph, for small values of the diameter bound D, calculation of $R_{\{s,t\}}(G, D)$ can be determined efficiently. The techniques introduced in this paper shed some light for the general case when planar graphs are under consideration.

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