Introduction of a new network reliability model to evaluate the performance of sensor networks

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Abstract— In this paper we present a new network reliability measure that is particularly useful to evaluate performance objectives of wireless sensor networks. A communication network can be modeled as directed graph $G = (V, E)$, composed of a set of nodes V, and a set of directed links E. Given that the links of the network underlying graph fail independently with known probabilities (nodes are perfectly reliable), and given a set K of *terminal* nodes (or participating nodes) and a distinguished terminal node s of K, the *K-terminal-to-sink* reliability measure, $R_{K,s}(G)$, is the probability of the event that the surviving links span a sub-digraph of G such that for each node u of K, there exists an operational directed path from u to s. In this paper we study a combinatorial property of graphs called the *domination* invariant which has been applied to efficiently compute the reliability of communication networks. Moreover we model wireless networks as random digraphs using current results in Information Theory and we discuss how the K-terminal-to-sink reliability could be applied to tackle several optimization as well as design problems in sensor networks.

Keywords—Domination invariant, reliability theory, sensor networks, outage probability, optimization theory.

I. INTRODUCTION

Failures in networks may arise from natural catastrophes, component wear out, or action of intentional enemies.

A communication network can be modeled by a digraph $G=(V, E)$, where V and E are the sets of nodes and links of G, respectively. Moreover the probabilities of failure of the components of a network could be represented by assigning probabilities of failure to the nodes and/or links of its underlying digraph. Several network reliability models have been studied to measure different reliability performance objectives of communication networks. In particular, in this paper, we will concentrate on one called the edge-reliability model, in order to present a clear description of its applicability to simulate real communication networks.

A *u,v-dipath* P between two nodes u and v of a digraph $G=(V,E)$ is defined as a sequence of links of G, $P = < (u_1=u,u_2), (u_2,u_3),\ldots, (u_{r-1},u_r=v) >$, where the nodes of P are distinct. Moreover the length of the u,v-dipath P is the number of links comprising the path (i.e., r-1). A directed cycle (*dicycle*) C of G is obtained from P by allowing $u=v$. Furthermore we say that a digraph G is *cyclic* if it contains a dicycle, otherwise is *acyclic.*

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Given a probabilistic directed topology $G=(V,E)$ with a distinguished set of terminal nodes K of V (also call participating nodes), a source node $s \in K$, and where the links fail independently with known probabilities (nodes are always operational), a widely studied network reliability measure is the classical source-to-K-terminal network reliability, $R_{s,K}(G)$ of G, which is defined as the probability that for each node $u \in K$, there exists an operational s,u-dipath spanned by the surviving links [1]-[2]. Unfortunately for general graphs the evaluation of the source-to-K-terminal reliability had been shown to be #P-complete [3].

Satyanarayana discovered a graph-theoretical invariant called the *signed-domination* of a digraph that plays an important role in significantly reducing the computational time for calculating the reliability of a network [1]-[2]. From that point on, the domination has been shown to be an important measure for evaluating the reliability of general systems [4]-[5]. Recent results involving the domination can be found in [6]-[7].

In this paper we introduce a new reliability measure, the K-terminal-to-sink network reliability that is particularly useful to assess the reliability of sensor networks. Indeed, given a probabilistic digraph $G=(V,E)$ with a distinguished set of terminal nodes K of V, a sink node $s \in K$, and where the links fail independently with known probabilities (nodes are always operational), let $R_{K,s}(G)$ be the probability that for each node $u \in K$, there exists an operational u,s-dipath spanned by the surviving links. The intuition behind this reliability model is clearly established by sensor networks where the sensor nodes, are represented by the nodes of the network underlying digraph, and each link of the digraph represents a wireless communication channel connecting a pair of sensor nodes. In a sensor network information gathered from a sensor node must be then transmitted to a gateway sensor (thru other nodes of the network), which is represented by the sink s of G. We are then assessing the probability that all the sensor nodes or a predefined set of sensor nodes (i.e., the terminal set K) can send information to the sink node s (via dipaths). Moreover we set the probability of failure of a wireless communication link (i.e., communication channel) equal to the power outage *Poutage* which was defined as the probability that the *capacity* of a channel is less than its *transmission rate* [8].

In Section II, we present some preliminary definitions and notation that will be used in the following sections, and introduce the domination for general systems. In Section III we give a complete characterization of the K-terminal-to-sink reliability domination of a digraph and we present an algorithm based on these theoretical results to calculate the reliability. In Section IV we represent a sensor network as a random digraph where the links represent the wireless communication channels connecting pairs of sensor nodes. Finally in Section V we present final conclusions and future research.

II. Preliminaries and the Domination

A. Preliminaries

As we are considering digraphs, we use the notation $ind_G(u)$ and $outd_G(u)$ to denote the *indegree* and *outdegree* of node u in G, where the indegree is the number of links directed into u and the outdegree is the number of links emanating from u.

The following notation will be used in the remaining of the paper where $G = (V, E)$ is a digraph with terminal node-set K, and distinguished sink node s of K:

a. Each link x is assigned an independent probability of failure $q(x) = 1 - p(x)$, where $p(x)$ is the probability of survival of x.

b. Let the sample space Ω represent the set of all possible subsets of E corresponding to sets of operational links.

c. Each subset H of Ω as probability occurrence $P(H) = \prod_{x\in H} p(x) \prod_{x\notin H} q(x)$.

d. H is a *pathset* if it contains a u,s-dipath for each $u \in K$.

e. Let $\theta_{K,s}(E) = \{H \epsilon\ \Omega\colon H \text{ is a pathset}\}$.

f. A pathset H is a minpath if for each $x \in H$, H-x is not a pathset of $\theta_{K,s}(E)$.

g. A K-tree $T = (V', E')$ of a digraph $G = (V,E)$ with terminal node-set K, is a tree covering all the nodes of K, such that any pendant node u (i.e., u has indegree 1 and outdegree 0, or indegree 0 and outdegree 1) of T belongs to K. In addition a K-tree T is a K,s-tree if $outd_T(s) = 0$ and there exists a unique u,s-dipath in T for each node u of T (see Fig. 1).

h. A digraph G is a K,s-digraph if every link of G lies in some K,s-tree of G.

i. From the definition of $R_{K,s}(G)$ and from definition e, one gets

$$R_{K,s}(G) = \sum_{H\in\theta_{K,s}(E)} \prod_{x\in H} p(x) \prod_{x\notin H} q(x) \qquad (1)$$

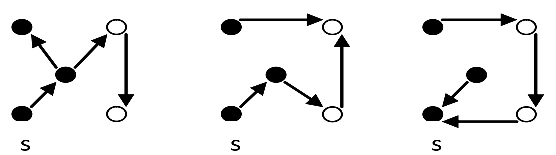

Fig. 1. Example of directed trees (darker nodes belong to K).

a) Tree not a K-tree, b) K-tree not a K,s-tree, c) K,s-tree.

The following lemma gives a characterization of the minpaths of $\theta_{K,s}(E)$ and it will established without a proof.

***Lemma* 1**. Let $G = (V, E)$ be a digraph with terminal set K and node $s \in K$; M is a minpath of $\theta_{K,s}(E)$ if and only if M is a K,s-tree of G.

Next we will discuss the definition of the domination invariant for the case of general systems.

B. Domination in general systems

A graph invariant called the reliability domination of a digraph G was introduced by Satyanarayana and Prabhakar for the classical network reliability measures [2]. The reliability domination plays an important role, allowing for efficiently implementing the principle of Inclusion-Exclusion of probability theory applied for evaluating the reliability of general systems.

Let E be a finite set, and 2^E be the power set of E. A nonempty subset $\mathcal{C} \subseteq 2^E$ is called a clutter of E if for any two elements $C_1, C_2 \,\epsilon\, \mathcal{C}$ whenever $C_1 \subseteq C_2$ then $C_1 = C_2$.

A pair $(E, \mathcal{C})$ will be referred to as a system and a system is *coherent* if each element of E is contained in some element of $\mathcal{C}$. A formation of $(E,\mathcal{C})$ is a collection of elements of $\mathcal{C}$ whose union yield E. The signed domination of the system $(E,\mathcal{C})$, and denoted as $d(E,\mathcal{C})$, is defined as the number of odd formations minus the number of even formations of E, where a formation is said to be odd or even if it is of odd or even cardinality, respectively. Trivially a system is *non-coherent* if it has no formations, so its signed domination is 0.

The clutters associated with the operation and failure of a specific element $x \,\epsilon\, E$ are defined as follows; let $\mathcal{C} - x = \{C - x\colon C \in \mathcal{C}\}$ and $\mathcal{C}_{-x} = \{C \,\epsilon\, \mathcal{C}\colon x \notin C\}$. Now $\mathcal{C}_{-x}$ is clearly a clutter but $\mathcal{C} - x$ may not be one. We define $\mathcal{C}_{+x}$ to be the collection of elements of $\mathcal{C} - x$ which are not proper supersets of some element of $\mathcal{C} - x$. For an element $x \in E$, $\mathcal{C}_{-x}$ and $\mathcal{C}_{+x}$ are called the minors with respect to x of $\mathcal{C}$.

Huseby (see [4], [5]) showed the following result:

***Theorem* 1**. If $(E,\mathcal{C})$ is a system, with $x \in E$, and minors $\mathcal{C}_{-x}$ and $\mathcal{C}_{+x}$ of $\mathcal{C}$, then

$$d(E,\mathcal{C}) = d(E - \{x\}, \mathcal{C}_{+x}) - d(E - \{x\}, \mathcal{C}_{-x}).$$

We look now at the case of the K-terminal-to-sink reliability of a digraph G=($V.E$). The system underlying our model is $(E,\mathcal{F})$ where E is the set of links of G, and where $\mathcal{F}$ is the collection of K,s-trees of G. A formation F of G is then a collection of K,s-trees whose union is E, the set of links of G. A formation is odd or even depending on whether F contains an odd or an even number of trees, respectively.

The signed domination of a digraph G=(V,E), simply called domination, and denoted as $d(E,\mathcal{F})$ with respect to a given subset $K \in V$, and node s of K, is the number of odd minus the number of even formations of G.

For brevity, in what follows we will use the standard notation $\mathcal{C}$ to represent $\mathcal{F}$, and we denote the domination $d(E,\mathcal{F})$ as $d(G)$. In addition we observe that if x is a link of G, T is a K,s-tree of G such $x \notin T$ if and only if T is a K,s-tree of G-x (i.e., the graph obtained from G by deleting link x). Therefore $d(E - \{x\}, \mathcal{C}_{-x}) = d(G - x)$.

Using this notation, the equation in Theorem 1 can be rewritten as

$$d(G) = d\,(E - \{x\},\ \mathcal{C}_{+x}\ ) - d(G - x). \qquad (2)$$

We next state the main results of this work, in which we characterize the domination of digraphs for the *K*-terminal-to-sink reliability model, and we discuss how these results can be used to efficiently compute the reliability of a network.

III. Characterization of the Domination and Algorithm

A. Domination

An alternative way (refer to (1)) of computing the reliability measure $R_{K,s}$ (*G*) of a probabilistic digraph *G*=(*V*,*E*) with terminal set *K* and distinguished sink node *s* of *K*, is by just considering the minpaths (i.e., the *K*,*s*-trees) of *G* and by means of the principle of Inclusion-Exclusion applied to probability theory.

For a digraph *G*=(*V*,*E*) with terminal set *K* and node *s* of *K*, let $\mathcal{M} = \{M_1, M_2, .., M_l\}$ be the set of all minpaths of $\theta_{K,s}(E)$ (i.e., *K*,*s*-trees). The situation where all the links of M_i operate (survive), is a random event which will be denoted by E_i. By Inclusion-Exclusion we obtain

$$R_{K,s}(G) = Pr\{\cup_{i=1}^{l} E_i\} = \sum_i Pr(E_i) - \sum_{i<j} Pr(E_i E_j) + \ldots + (-1)^{l+1} Pr(E_1 E_2 \ldots . E_l), \quad (3)$$

where the event $E_1 E_2 \ldots E_m$ is the event that all the links of the subgraph obtained by the union of $M_1, M_2, ..,$ and M_m, are operating.

In (3) the terms correspond to sub-digraphs of *G* obtained by the union of minpaths. As discussed previously, in the *K*-terminal-to-sink reliability of a digraph *G*=(*V*,*E*) with terminal set *K*, and node *s* ∈ *K*, the minpaths are the *K*,*s*-trees of *G*, and the subgraphs obtained by the union of minpaths, are the *K*,*s*-digraphs. The same *K*,*s*-digraph can be obtained from different formations, this means that it may appear in (3) more than once, sometimes with positive sign, and sometimes with negative sign, depending if the corresponding formation has an odd or an even number of *K*,*s*-trees. In fact, its net contribution will be exactly the number of odd minus the number of even formations of the *K*,*s*-diagraph, that is, its domination invariant. Thus if $\mathcal{H}$ is the set of *K*,*s*-diagraphs of *G*, then (1) or (3) we can be re-written as

$$R_{K,s}(G) = \sum_{H \in \mathcal{H}} d(H) \Pr(H), \quad (4)$$

where *Pr* (*H*) is the probability that the links comprising *H* are operative (i.e., the product of the probabilities of survival of its links).

In the remaining of the section we will characterize the domination of *K*,*s*-digraphs and we will give a simple rule to calculate the domination. If *H* has a dicycle then its domination is 0; otherwise if it is acyclic then its domination is $(-1)^{e(H)-n(H)+1}$, where *e*(*H*) and *n*(*H*) are the number of links and nodes of *H*, respectively. Interesting enough this characterization coincides with the one obtained for the source-to-*K*-terminal reliability obtained by Satyanarayana [1]-[2]. This simple characterization of the domination for this reliability model yields an efficient algorithm for the calculation of $R_{K,s}(G)$, especially if *G* has several dicycles, even though determination of $R_{K,s}(G)$ may still be a *#P*-complete problem.

The following lemma plays an important role in the characterization of the domination.

***Lemma* 2**. Let *G*=(*V*, *E*) be a digraph with terminal set *K*, and distinguished node *s* ∈ *K*. Suppose that *x* = (*u*, *s*) is a link of *G* for some node *u* of *V* and $outd_G(u)$ > 1, then $d(G) = -d(G-x)$.

Proof. As $outd_G$ *(u)* > 1, let *x'*= (*u*, *v*), $v \neq s$, and suppose *T'* is a *K*,*s*-tree of *G* such that *x'* is a link of *T'*. Considering (2) we will show that $d\ (E-\{x\},\ \mathcal{C}_{+x}) = 0$, or equivalently that that the system $(E-\{x\}, \mathcal{C}_{+x})$ has no formations (i.e., set of *K*,*s*-trees such that their union yields *E* – {*x*}). We first note that *T'* is a *K*,*s*-tree of $\mathcal{C} - x$. Since *T'* is a *K*,*s*-tree, from definition g of Section II, there exists exactly one *u*,*s*-path (not containing *x*), and by deleting *x'* and adding *x* we still have a unique *a*,*s*-dipath for every node *a* of *T'*; but we may have created some pendant nodes (i.e., nodes of indegree 0) that do not belong to *K*. Let *T* be the tree obtained from *T'* by adding *x*, deleting *x'* and possibly some pendant nodes that do not belong to *K*. Thus *T* is a *K*,*s*-tree and $T - x \subseteq T' - x' \subset T'$, but both *T* - *x* and *T'* belong to $\mathcal{C} - x$, and *T'* is a proper superset of *T* - *x*, consequently $T' \notin \mathcal{C}_{+x}$. This implies that not element of $\mathcal{C}_{+x}$ contains *x'* thus the system $(E-\{x\}, \mathcal{C}_{+x})$ is non-coherent, and $d(E-\{x\}, \mathcal{C}_{+x}) = 0$. Then from (2) the lemma follows.■

The following lemma plays an important role in the characterization of the domination.

***Lemma* 3**. Let *G* = (*V*, *E*) be a digraph with terminal set *K*, and distinguished node *s* ∈ *K*. Suppose that *x*=(*s*, *u*) is a link of *G* for some node *u* of *V*, then *G* is not a *K*,*s*-digraph and $d(G) = 0$.

Proof. The link *x* does not lie in any *K*,*s*-tree of *G*, thus the system $(E, \mathcal{C})$ has no formations, and *d*(*G*)=0. Moreover *G* is not a *K*,*s*-digraph. ■

We define the following operation on a digraph *G*= (*V*,*E*) with terminal set *K* and node *s* ∈ *K*.

- **OP** (*G*, *K*, *s*): Let $V' = \{u \in V - s : (u,s) \in E\}$, and suppose $\forall\, u \in V',\ outd_G(u) = 1$. This operation returns a digraph *G** with terminal set $K^* = K - V' - \{s\} \cup \{s^*\}$, where *G** is obtained from *G* by identifying *s* and the nodes of *V'* into a node *s** (i.e., *s** is the new sink node of *G**).

***Lemma* 4**. Let *G* = (*V*, *E*) be a *K*,*s*-digraph with terminal set *K*, and distinguished node *s* ∈ *K*, and suppose that for every node *u* of *G* such (*u*, *s*) is a link of *G*, $outd_G$ (*u*) = 1. Suppose that *G** is the digraph returned by **OP** (*G*, *K*, *s*), then *d* (*G*) = *d* (*G**). Moreover *G** is also a *K**,*s**-digraph.

Sketch of Proof. We must show the following:

1) There exists a one-to-one correspondence between the *K*,*s*-trees of G and the *K**,*s**-trees of *G**.
2) There exists a one-to-one correspondence between the formations of *G* and the formations of *G**; moreover a formation of *G* and its corresponding formation of *G** have the same cardinality.

The proofs of correspondences 1) and 2) are left for the readers, by considering the following correspondence: let $U = \{u_1, u_2, \ldots, u_r\}$ be the set of nodes of *G* such that (u_i, s) is a link of *G*. Given a *K*,*s*-tree *T* we construct a *K**,*s**-tree *T'* by identifying *s* and the nodes adjacent to *s* in *T* into a node s^*. However the node s^* is the sink node of *T'* and *T'* terminal node-set is $K^* = K - U - \{s\} \cup \{s^*\}$. ■

The following lemma is concerned with cyclic *K,s*-digraphs.

***Lemma* 5**. Let *G* = (*V*, *E*) be a cyclic *K,s*-digraph with terminal set *K*, and distinguished node *s* ∈ *K*, then

1) Suppose that *x*= (*u*,*s*) is a link of *G*, and $outd_G$ (*u*) > 1, then *G*-*x* is also cyclic.
2) Suppose that *G** is the digraph returned by **OP** (*G*, *K*, *s*) with terminal node-set *K**, and distinguished node s^*, then *G** is also cyclic.

Proof. To prove 2) let $U = \{u_1, u_2, \dots, u_r\}$ be the set of nodes such that (u_i ,*s*) is a link of *E*. As *G* is a *K,s*-digraph then by Lemma 3, *G* does not contain a link (*s*, *v*), $v \in U$, and as the nodes of *U* have outdegree 1, then the nodes of a dicycle $C =< (v_1, v_2), (v_2, v_3), .., (v_m, v_1) >$ cannot intercept $U \cup \{s\}$; then when *s* and the set *U* of *G* are identified into a single node s^* to yield *G**, the same dicycle remains in *G**.

To prove 1), as from Lemma 3 since *G* is a *K,s*-digraph then *G* does not contain a link (*s*,*v*) for some node *v* of *G*, therefore no dicycle can contain a link *x*= (*u*,*s*), therefore *G*-*x* is also cyclic. ■

An analogous lemma can be stated regarding acyclic digraphs.

***Lemma* 6**. Let *G* = (*V*, *E*) be an acyclic *K,s*-digraph with terminal set *K*, and distinguished node *s* ∈ *K*, then

1) Suppose that *x* = (*u*,*s*) is a link of *G*, and $outd_G$ (*u*) > 1, then *G*-*x* is also an acyclic *K,s*-digraph.
2) Suppose that *G** is the digraph returned by **OP** (*G*, *K*, *s*) with terminal node-set *K**, and distinguished node s^*, then *G** is also acyclic.

Proof. To prove 2) we will proceed by contradiction. Let $U = \{u_1, u_2, \dots, u_r\}$ be the set of nodes such that (u_i, *s*) is a link of *E*. Suppose that the identification of *U* and *s* into a node s^* creates a dicycle $C =< (v_1, v_2), (v_2, v_3), \dots, (v_m, v_1) >$ in *G**. It must be the case then that $s^*=v_i$, where v_i is a node of the dicycle *C*, thus either there exists a link (*s*, v_{i+1}) or there exists a link (u_j , v_{i+1}) in *G*, where u_j is a node of *U* and where v_{i+1} is a node of *C* and of *V*-*U*-{*s*}. This implies that either the outdegree of *s* is at least 1 or the outdegree of u_j is at least 2. By Lemma 3, since *G* is *K,s*-digraph then outdegree of *s* in *G* is 0, and under the assumption that outdegree of u_j is 1, then a contradiction is reached.

To prove 1) it is trivial to show that *G* - *x* is acyclic as deletion of a link cannot create a dicycle. Next we want to show that every link of *G* – *x* is covered by some *K,s*-tree. Let *x'* = (*u*, *v*) be a link (under the assumption of the existence of the link *x* = (*u*, *s*)). As *G* is a *K,s*-digraph then there exists a *K,s*-tree *T'* of *G* containing *x'*; but *T'* is also a *K,s*-tree of *G*-*x* as well. Suppose that *T* is a *K,s*-tree of *G*, such as *x*=(*u*,*v*) lies in this tree. Here we want to show that we can find a *K,s*-tree in *G*-*x* that contain the links of *T* except for *x* itself, consequently forcing *G*-*x* to be a *K,s*-digraph. The deletion of *x* from *T* yields two trees, T_1 and T_2 with terminal sets K_1 and K_2, respectively, where $K_1 \cup K_2 = K$. Moreover *u* and *s* belong to different trees; without loss of generality let *u* belong to T_1 and *s* to T_2. From the definition of a *K,s*-tree (definition g of Section II-A), there exists a dipath *a,s*-dipath for each node *a* of *T*, thus T_1 is a K_1,*u*-tree (where *u* may or may not belong to K_1) and where T_2 is a K_2,*s*-tree. Next it is easy to show that there exists a *u,s*-dipath $P =< (u, v), (v, v_1), .., (v_m, s) >$ in *G* not containing the link *x*=(*u*,*s*), as *G* is a *K,s*-digraph, and, consequently, there exists a *K,s*-tree *T'* containing *x'*. Next we realize that the path *P* does not intercept any node *a* of T_1, otherwise, as T_1 is a K_1,*u*-tree, this tree contains an *a,u*-dipath, together with the *u,a*-dipath of *P*, will create a dicycle in *G*. Thus *P* must first intercept a node *b* of T_2 as a dipath from *u* to *s* exists in *G*. Let $P' =< (u, v), (v, v_1), .., (v_r, b) >$ be such sub-dipath of *P*, and consider the tree *T** formed by the union of T_1, T_2 and *P'*; this tree contains the links of *T*-*x*. As T_1 is a K_1,*u*-tree and T_2 is a K_2,*s*-tree, *T** is a *K,s*-tree of *G*-*x*. ■

Taking into account the previous lemmas, the following two theorems give a complete characterization of the domination for digraphs *G* = (*V*,*E*) with terminal set *K*, and distinguished node $s \in K$.

***Theorem* 2**. Let *G* = (*V*, *E*) be a cyclic digraph with terminal set *K*, and distinguished node $s \in K$, then *d* (*G*)=0.

Proof. If *G* is not a *K,s*-digraph (i.e., it contains a link not covered by some *K,s*-tree) then its domination is 0, and the system $(E, \mathcal{C})$ has no formations, therefore the theorem follows trivially. Thus we can assume that *G* is a cyclic *K,s*-digraph.

We will proceed by induction on the number of links |*E*|.

Basis: Let |*E*|=0. Since there is not cyclic *K,s*-digraphs with no links, then the assertion is vacuously true.

For the induction step, let *G* let be a cyclic *K,s*-digraph with |*E*| > 0, with terminal set *K*, and distinguished node $s \in K$, and suppose the theorem holds for all cyclic *K',s'*-digraphs with fewer links than *G*.

Suppose that in *G* there exists a link *x* = (*u*,*s*), such as $outd_G$ (*u*) > 1, for some node *u* of *G*.

If *G*-*x* is not a *K,s*-digraph then *d*(*G*-*x*)=0, and by Lemma 2, *d*(*G*)= - *d*(*G*-*x*), thus *d*(*G*)=0.

If *G*-*x* is a *K,s*-digraph, then by Lemma 2, *d*(*G*)=-*d*(*G*-*x*), but by Lemma 5-1, *G* - *x* is also cyclic and it has fewer links than *G*, thus by the induction hypothesis *d*(*G*)=0.

Suppose next that every node adjacent to *s* in *G* has outdegree 1, and let *G** be the digraph with terminal set *K** and node s^* obtained from operation **OP** (*G*, *K*, *s*); since *G* is a *K,s*-digraph, by Lemma 4, *G** is *K*,s**-digraph, moreover *d*(*G*)=*d*(*G**). But from Lemma 5-2, *G** is cyclic as well, thus from the induction hypothesis, as *G** has fewer links than *G*, then *d*(*G*) = *d*(*G**)=0.■

***Theorem* 3**. Let *G*= (*V*, *E*) be an acyclic *K,s*-digraph with terminal set *K*, and distinguished node $s \in K$, then $d(G) = (-1)^{e-n+1}$, where *n*=|*V*|, and *e*=|*E*|.

Proof. We will proceed by induction on the number of links |*E*|. Basis: Let |*E*|=0. The only acyclic *K,s*-digraph with |*E*|=0 is $G = (\{s\}, \emptyset)$, and $d(G) = 1 = (-1)^{e-n+1}$.

For the induction step, let *G* let be an acyclic *K,s*-digraph with |*E*| > 0, with terminal set *K*, and distinguished node $s \in K$, and suppose the theorem holds for all acyclic *K',s'*-digraphs with fewer links than *G*.

Suppose that in G there exists a link x = (u,s), such as $outd_G(u) > 1$, for some node u of G By Lemma 6-1, $G - x$ is also an acyclic K,s-digraph, and it has e'=|E|-1 links and n=|V| nodes, thus by the induction hypothesis $d(G-x) = (-1)^{e-1-n+1} = (-1)^{e-n}$. Then from Lemma 2, it follows that $d(G) = -d(G-x) = -(-1)^{e-n} = (-1)^{e-n+1}$.

Suppose next that every node adjacent to s has outdegree 1, and let G^* be the digraph with terminal set K^* and node s^* obtained from operation **OP** (G, K,s); since G is a K,s-digraph, by Lemma 4, so is G^*, and $d(G)=d(G^*)$. Let U be the set of nodes of outdegree 1 adjacent to s in G; the number of links of G^* is $e^*=e-|U|$, and the number of nodes is $n^*=n-|U|$, where $e=|E|$, and $n=|V|$ (these equalities are determined by the fact that each node u of U is only adjacent to s, by a link (u,s), and since G is a K,s-digraph then it doesn't exist a link (s,u) in G). But from Lemma 6-2, G^* is acyclic as well, thus from the induction hypothesis, as G^* has fewer links than G, $d(G) = d(G^*) = (-1)^{e*-n*+1} = (-1)^{(e-|U|)-(n-|U|)+1} = (-1)^{e-n+1}$; thus the Theorem follows. ■

By (4), the calculation of the reliability is limited to just detecting the K,s-digraphs of G, and from Theorem 2 and Theorem 3, it is clear that from the K,s-digraphs, only the acyclic ones play a role with a positive or negative contribution to the reliability. Indeed, for each acyclic K,s-digraph with n nodes and e links, its contribution is $(-1)^{e-n+1}$ times the product of the probabilities of survival of its links. This result yields an efficient way to calculate the reliability, even though the general problem of evaluating the reliability may be still a #P-complete problem. These simplifications will be captured by an algorithm to evaluate the K-terminal-to-sink reliability, which will be presented in the next subsection.

B. Algorithm

In this section we present an algorithm for the computation of the K-terminal-to-sink reliability of a digraph G, based upon (4) and the characterization of the domination of its K,s-digraphs, as stated by Theorem 2 and Theorem 3.

It is easy to see that in any digraph G containing a set of parallel links $\{x_1, x_2, \dots, x_m\}$ emanating from a node u and directed into a node v, with corresponding reliabilities $\{p(x_1), p(x_2), \dots, p(x_m)\}$, can be replaced by a single link $x = (u, v)$ with reliability $p(x) = 1 - \prod_{i=1}^{m}(1 - p(x_i))$, without actually affecting the reliability of G; thus we are only concerned with digraphs with not parallel links.

We need to define when a digraph is K,s-connected.

Definition 1 For a digraph G = (V, E), with terminal set $K \subseteq V$, and distinguished node $s \in K$, we say that G is K,s-connected if there exists in G an u,s-dipath for every $u \in V$.

If $outd_G(s) = 0$, we will denominate this digraph s-sinked, and from this point on we will be concerned only with s-sinked digraphs, since if that is not the case, then $d(G) = 0$, as stated by Lemma 3.

We next need to define irrelevant links.

Definition 2 Given a graph G = (V,E), with terminal set $K \subseteq V$, node $s \in K$, a link $x = (u, v) \in E$ is an irrelevant link if:

a) The link x belongs to a connected component G'=(V',E') of G, where $V' \subseteq V - K$.

b) The node $u \in V - K$ has $ind_G(u) = 0$.

c) The node $v \in V - K$ has $outd_G(v) = 0$.

According to Theorem 2 and Theorem 3, an algorithm to compute $R_{K,s}(G)$ should be only concerned in just identifying acyclic K,s-digraphs. The following theorem gives a sufficient condition for such digraphs and its proof is left for the reader.

Theorem 4. Given a digraph G=(V,E) with terminal set K, and node $s \in K$, suppose that G is an acyclic, K,s-connected digraph, with no irrelevant links, then G is a K,s-digraph.

We present next an algorithm for efficiently generating precisely all these digraphs having non-null domination.

As a first step, we assume that G is s-sinked. If this is not the case we can simply delete all links emanating from s, obtaining a s-sinked digraph. Moreover parallel links are replaced by a single link whose reliability is obtained as explained at the beginning of this section.

The algorithm has four stages:

a) If G has irrelevant links, generate a sub-digraph from G by deleting these links and any isolated node $u \in V - K$ obtained from this deletion.

b) If G is not K,s-connected, then do not generate any subgraphs from G.

c) If G is K,s-connected and contains a dicycle, generate acyclic subgraphs of G.

d) If G is K,s-connected, acyclic, then generate all possible subgraphs of G.

Generation of duplicate subgraphs at all stages is completely avoided by a simple check.

The algorithm grows a rooted directed tree with the following properties:

- Vertices represent non-empty subgraphs of G, the root vertex being G itself. Any vertex, say r, corresponds one-to-one with the subgraph G_r, which is of one of the following four types: a) G_r contains irrelevant links, b) G_r is not K,s-connected, c) G_r is K,s-connected and cyclic, d) G_r is K,s-connected and acyclic.
- An arc directed from vertex i to vertex j of the tree is labeled X, where X represents the set of links deleted from G_i to obtain G_j.

Consider the following additional definitions:

I. *Father* (*Child*): Vertex i (j) is the father (child) of j (i) if there exists an arc directed from i to j.

II. *Ancestor*: Vertex i is the ancestor to j when i is contained in the path from the root vertex to j ($i \neq j$).

III. *Brother*: Vertices having the same father are termed brothers.

IV. *Younger (Elder) Brother*: A vertex i is the younger (elder) brother of vertex j, if the algorithm generates the children of vertex i later (earlier) than the children of vertex j.

The algorithm starts at the root vertex and grows the tree progressively. There are four rules for generating the children of vertex *r*, depending on the nature of G_r.

Rule 1 G_r *has irrelevant links*. Let X' be the label corresponding to the set of irrelevant links of G_r. In this case generate a new node representing the digraph obtained from G_r by deleting these links (and possibly any isolated nodes obtained from this deletion), provided $X' \cap X = \emptyset$, where *X* is the label of the arc incident into the elder brothers of *r* or elder brothers of an ancestor of *r*; otherwise do not generate any children from G_r.

Rule 2 G_r *is not K,s-connected*. In this case G_r does not generate any children.

Rule 3 G_r *is K,s-connected and cyclic*. Consider a dicycle *C* in G_r containing the set of links $\{x_1, x_2, \ldots, x_c\}$. Then $G_{r_j} = G_r - x_j$, $(j = 1,2,\ldots,c)$, is a child of G_r, provided $\{x_j\} \cap X = \emptyset$, where *X* is the label of an arc incident into the elder brothers of *r*, or elder brothers of an ancestor of *r*. Determination of a dicycle is determined for example by application of Depth First Search (DFS). Clearly a state $G_r - x_j$ where x_j does not belong to the dicycle *C*, contains also *C* thus by Theorem 2, $d(G_r - x_j) = 0$, so it is not necessary to generate this state.

Rule 4 G_r is *K,s*-connected, and acyclic. Let $G_r = (V_r, E_r)$, and G_r does not have irrelevant links; it follows from Theorem 4 that G_r is a *K,s*-digraph, therefore contributing to the reliability by $(-1)^{|E_r|-|V_r|+1}\prod_{e\in E_r} p(e)$. Moreover let $G_{r_j} = G_r - x_j$ be a child of G_r, provided $\{x_j\} \cap X = \emptyset$, where *X* is the label of an arc incident into any elder brother of *r*, or elder brother of an ancestor of *r*.

The algorithm applies the previous rules recursively, and employs a rooted tree, called *Auxt*, as an auxiliary data structure that is used to maintain the states already generated and to avoid state duplications (i.e., at each step, a check is performed to see if the links to be deleted from a digraph are contained in the label of an arc incident into any brother, or elder brother of an ancestor of this digraph). We now present the pseudo-code of the algorithm:

Algorithm

Input: Original *s*-sinked digraph *G*.

Output: *K*-terminal-to-sink reliability $R_{K,s}(G)$.

Data structures:

- $\mathcal{P}(E)$. Represents the operational probabilities of the set of links *E* of the original digraph *G*, and the operational probability of a link $x \in E$ denoted as *p*(*x*).
- ***R***. Global variable to represent *K*-terminal-to-sink reliability. Originally *R*=0.
- ***r***. Current vertex being considered. This is a global variable and originally *r*=0.
- G_r. Current digraph under consideration. Originally $G_0=G$.
- n_r, e_r. Number of nodes of G_r and number of links of G_r.
- ***Auxt***. Rooted tree auxiliary data structure. Originally *Auxt* contains only the vertex *r*=0, representing the original graph.

Auxiliary Procedures:

- *AddAuxt* (vertex *l*, vertex *m*, label *X*). This procedure will add an arc from vertex *l* into a new vertex *m* of *Auxt*, whose label *X* corresponding to a set of links deleted from G_l to obtain G_m.
- *bool CheckAuxt* (vertex *l*, label *X*). This procedure will backtrack from vertex *l* to find if any of the links represented by the label *X* is a link of the set of links corresponding to a label incident into any elder brother or ancestor's elder brother of a vertex *l* (we assume that each vertex contains the label of its father). If that is the case it will return *true*, otherwise it will return *false*. This routine is computationally efficient, since the longest possible path from the root of *Auxt* is of at most |*E*| arcs.

```
CalcRel (Graph Gr)
Begin
1. Let crntvrtx=r; current vertex of the rooted tree;
2. If Gr = (Vr,Er) contains a set of irrelevant links E';
     Let X be the label of E';
     If CheckAuxt (crntvrtx, X) = false;
        Let r=r+1;
        AddAuxt (crntvrtx, r, X);
        Let Gr =Gcrntvrtx-E';
        CalcRel (Gr);
     End-if
     return;
   End-if
3. Apply Depth First Search to determine
   K,s-connectedness or to detect dicycles.
4. If Gr = (Vr, Er) is not K,s-connected return;
5. If Gr = (Vr, Er) is cyclic;
     Let C = {x1, x2, ..., xc } be the links of a dicycle of Gr.
     For (xi ∈ C) do
        If CheckAuxt (crntvrtx, X=xi) = false;
           Let r=r+1;
           AddAuxt (crntvrtx, r, xi);
           Let Gr =Gcrntvrtx-xi;
           CalcRel (Gr);
        End-if
     End-for
     return;
   End-if
6. //acyclic K,s-digraph
     R = R + (-1)^(|Er|-|Vr|+1) ∏_{e∈Er} p(e).
     For (xi ∈ Er) do
        If CheckAuxt (crntvrtx, X=xi) = false;
           Let r=r+1;
           AddAuxt (crntvrtx, r, xi);
           Let Gr =Gcrntvrtx-xi;
           CalcRel (Gr);
        End-if
     End-for
     return;
End
```

Of the possible $2^{|E|}$ states (i.e., digraphs) to be evaluated, steps 2, 4, and 5 of the above algorithm represent a significant reduction on the total number of executable operations performed, since many states are avoided, especially when the digraphs contain irrelevant links, or when they contain several directed cycles, or the digraphs are not *K,s*-connected.

C. *Example for the application of algorithm*

Consider the graph G depicted in Fig. 2. First we delete any link emanating from the sink node s, as explained in section III-B (i.e., links e_1 and e_3). Next parallel links e_4 and e_5 are replaced by a single link $e_{4,5}$ with reliability $p(e_{4,5})=1-(1-p(e_4))(1-p(e_5))$, obtaining the digraph G_0. Then DFS is applied

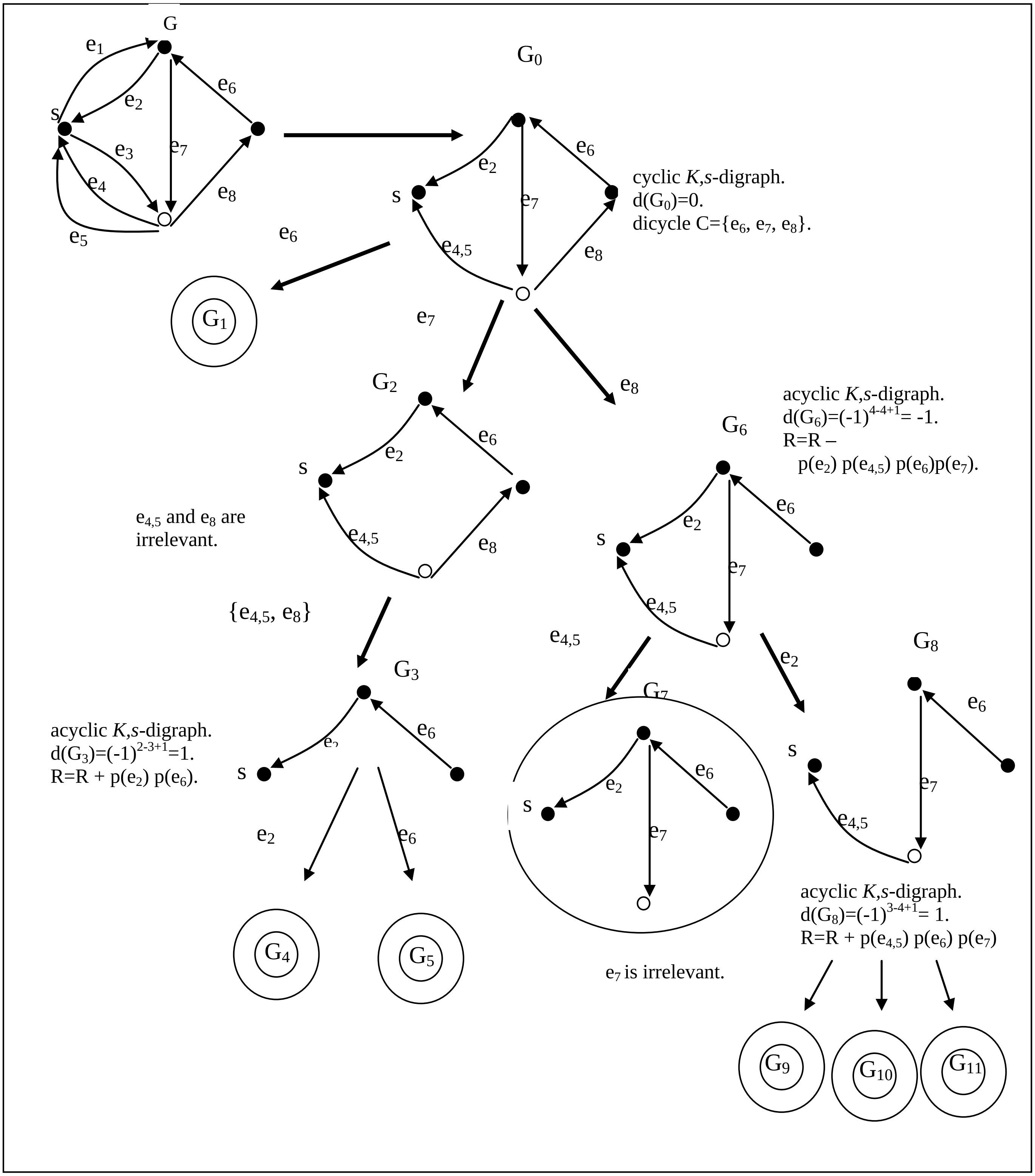

Fig. 2. Application of algorithm **CalcRel** to evaluate the reliability of a digraph G (terminal nodes of K are the black nodes).

to G_0 and the dicycle C={e_6, e_7, e_8} is detected generating the subgraphs G_1, G_2, and G_6, by deleting e_6, e_7, and e_8 from G_0, respectively. G_1 is not a *K,s*-digraph (double circled shape) thus it does not generate any children. Next, G_2 contains two irrelevant links ($e_{4,5}$ and e_8) which are deleted (and resultant isolated node) to generate G_3 which is an acyclic *K,s*-digraph with domination 1, thus contributing to the total reliability; as G_3 is a *K,s*-tree, any subgraph generated from it (i.e., G_4 and G_5) is not a *K,s*-digraph. Next G_6 is an acyclic *K,s*-digraph with domination -1, thus the product of the reliabilities of its links is deducted from the total reliability; in addition G_6 generates two children, G_7 and G_8, by deleting links $e_{4,5}$ and e_2, respectively. Subgraphs of G_6 obtained by the deletion of e_6 and e_7 are not generated since G_6 has elder brothers G_1 and G_2, which were generated by deleting e_6 and e_7 from G_0, respectively. Next G_7 has an irrelevant link e_7 but it is not deleted from G_7 to generate a child because e_7 is the label incident into an ancestor's elder brother (i.e., G_2) of G_7. G_8 is an acyclic *K,s*-digraph with domination 1, therefore contributing to the total reliability by the product of the reliabilities of its links. Finally the children of G_8 are not *K,s*-digraphs (double circled shapes) thus they do not generate any children.

IV WIRELESS NETWORK MODELING AND SENSOR NETWORKS

A. Random Graphs and link failure representation

Wireless transmissions are degraded by a phenomenon known as the *Path Loss*, which is known as the difference between the transmitted power p_o and the received power of a signal p_r. The relationship between p_r and p_o is given by the relation $p_r = p_0 / d^n$, where d is the distance between the transmitter and receiver and n is a constant between 2 and 4, known as the *path loss exponent*, representing the degradation of the signal due to the physical characteristics of the terrain embracing the communication between the transmitter and receiver.

Shannon's law provided the theoretical maximum rate at which error free digits can be transmitted [9]. Mathematically, the capacity of a communication channel C is defined by the relation $C = b\log_2(1+SNR)$, where b is the bandwidth in Hz, and SNR is the signal to noise ratio at the receiver end. The noise η is a zero mean additive white Gaussian noise with average power of σ_η^2, and assuming a transmitted input signal x, the SNR is then defined as $|x|^2/\sigma_\eta^2$.

A more representative model was presented for the capacity of a wireless communication channel in [8]. In this model, the instantaneous capacity of a wireless link is treated as a random variable and it is represented by the relation

$$C = \log_2(1 + |f|^2 / d^n \, SNR), \quad (5)$$

where f is the fading state of the channel modeled as a Rayleigh random variable. The probability of a link (communication channel) power outage *Poutage* (or link failure probability) was then defined as the probability of the event that the channel capacity C is less than the transmission rate R usually expressed in bits per channel use (i.e., *Poutage* = *Pr* {C<R}); the outage probability of a communication link is then given by

$$Poutage = 1 - \exp\left(\frac{-d^n}{\mu SNR'}\right), \quad (6)$$

where $\mu = E(|f|^2)$ [8]. The probability of a successful reception, or equivalently the *link reliability*, for a Rayleigh fading link with fixed distance is then given by the following simple expression

$$Psuccess = 1 - Poutage = \exp(-d^n / \mu SNR'). \quad (7)$$

The importance of this link reliability representation is that the communication of real wireless networks can be then accurately simulated using standard Graph Theoretical models, and therefore several communication optimization problems can be successfully tackled.

B. Sensor Networks

A wireless sensor network (WSN) consists of spatially distributed autonomous sensors to cooperatively monitor physical or environmental conditions, such as temperature, sound, vibration, pressure, motion or pollutants. WSNs are now used in many industrial and civilian application areas, including industrial process monitoring and control, machine health monitoring, environment and habitat monitoring, healthcare applications, home automation, and traffic control.

In addition to one or more sensors, each node in a sensor network is typically equipped with a radio transceiver or other wireless communications device, a small microcontroller, and an energy source, usually a battery. A sensor network normally constitutes a wireless ad-hoc network, meaning that each sensor supports a multi-hop routing algorithm where nodes function as forwarders, relaying data packets to a base station. Typically in a sensor network each node transmits information to a sink node (gateway sensor node) via the other sensor nodes constituting the network; the gateway node then relays information to a base station connected to it (see Fig. 3); further research results on sensor networks can be found in [10]-[15].

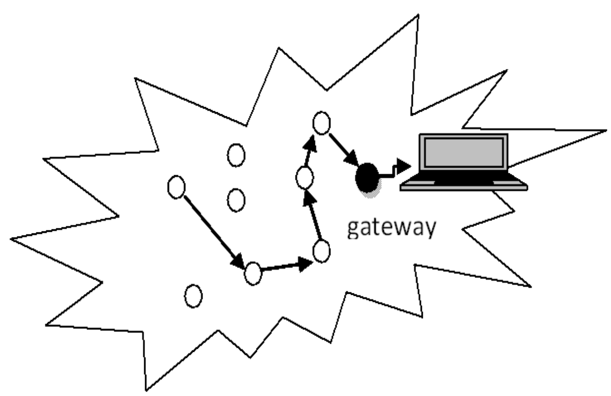

Fig. 3. Multihop wireless sensor network architecture.

Since each node of the network can transmit as well as to receive signals, we could represent this sensor network as a digraph. The communication between two nodes i and j is represented by two anti-parallel links, where the link (i, j) represents the transmission from transmitter i to receiver j; similarly we define the link (j, i) but by interchanging the roles of the nodes. A link $l=(i, j)$ has failure probability $q(l)$ as stated by (6). Also in this model it is very possible that $q((i, j)) \neq q((j, i))$ since for example the nodes may have different receiving and/or transmitting characteristics.

Let $G = (V,E)$ be a digraph with node-set V and link-set E. Moreover let s be the predefined sink-node (i.e., gateway node) and K be the terminal node-set of G. Then $R_{K,s}(G)$, which was introduced and discussed in the previous sections, gives the probability of the event that the sink s will be able to received information (via directed paths) from every participating node in K. In particular $R_{u,s}(G)$ measures the probability that a sensor node u will be able to send information to the sink node s of G and it will substitute the notation $R_{K,s}(G)$, $K=\{u, s\}$. Using this new reliability measure several optimization as well as network design problems in sensor networks could be tackled. As an example if we let K be the set of crucial sensor nodes, we could partition the set V-K into p subsets depending on how the K-terminal-to-sink reliability decreases when the nodes of a subset are put to sleep for a period of time.

V Conclusions

We have introduced a new network reliability measure, the K-terminal-to-sink reliability and we've presented an efficient algorithm for its calculation, based upon a combinatorial parameter called the domination invariant which has been studied, by many researchers, to efficiently compute the reliability of general systems. This new reliability model is particularly useful to assess performance objectives of sensor networks which can be modeled as random digraphs, and where, a directed link of the digraph represents a communication channel between a transmitting sensor node and a receiving sensor node. Moreover a communication link was represented by a random variable using current results in Information Theory applied to wireless communications; the K-terminal-to-sink reliability measure can be applied to solve optimization problems in communication theory, and to address design problems as well.

Future research will involve the development of new algorithms to address the above mentioned problems in sensor networks, based upon the new reliability parameter.

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