Diameter-related properties of graphs and applications to network reliability theory

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Abstract: Given an undirected graph $G = (V, E)$, two distinguished vertices s and t of G, and a diameter bound D, a D-s,t-path is a path between s and t composed of at most D edges. An edge e is called D-irrelevant if does not belong to any D-s,t-path of G. In this paper we study the problem of efficiently detecting D-irrelevant edges and also study the computational complexity of diameter-related problems in graphs. Detection and subsequent deletion of D-irrelevant edges have been shown to be fundamental in reducing the computational effort to evaluate the Source-to-terminal Diameter-Constrained reliability of a graph G, $R_{\{s,t\}}(G, D)$, which is defined as the probability that at least a path between s and t, with at most D edges, survives after deletion of the failed edges (under the assumption that edges fail independently and nodes are perfectly reliable). Among other results, we present sufficient conditions to efficiently recognize irrelevant edges and we present computational results illustrating the importance of embedding a procedure to detect irrelevant edges based on these conditions, within the frame of an algorithm to calculate $R_{\{s,t\}}(G, D)$, built on a theorem of Moskowitz. These results yield a research path for the theoretical study of the problem of determining families of topologies in which $R_{\{s,t\}}(G, D)$ can be computed in polynomial time, as the general problem of evaluating this reliability measure is NP-Hard.

Key–Words: Network reliability, diameter constraint, paths, factoring, topological reductions.

1 Introduction

Unless otherwise stated, in this paper we consider undirected graphs $G = (V, E)$, where V represents a finite set of vertices and E is a finite set of edges.

The purpose of this work is two-fold. We first investigate, from a computational point of view, diameter properties of graphs related to the following optimization problem: given two vertices s and t of G, we would like to efficiently identify edges that do not belong to any path between s and t of length less or equal to a given bound D; we then apply some of the results shown to compute more efficiently the Diameter-Constrained reliability (*DCR*) of a communication network (originally introduced in [17]), a constrained version of the classical network reliability measure (refer to [20, 21, 22, 24, 25] for further discussion on this classical model). This study serves as a guide to address the problem of identifying families of topologies in which the *DCR* can be computed in polynomial time (as its computation is known to be NP-Hard).

Given a probabilistic graph $G = (V, E)$, a set of terminal nodes $K \subseteq V$, and a diameter bound D, in which each edge $e \in E$ has been assigned a probability of failure $q_e = 1 - r_e$ (r_e is called the reliability of the edge e) under the assumption that edges fail independently and nodes are perfectly reliable, the *K-terminal Diameter-Constrained* reliability, $R_K(G, D)$, gives the probability of the event that for each pair of nodes $x, y \in K$, a path between x and y of length (i.e., number of edges comprising the path) D or less, called a D-x,y-path, survives after deletion of the failed edges. In this paper we consider the case when $K = \{s, t\}$, known as the Source-to-terminal Diameter-Constrained reliability of a graph G, denoted as $R_{\{s,t\}}(G, D)$. For the classical reliability measure, the K-terminal reliability $R_K(G)$, of a graph G, is the probability that after the removal of the failed edges, each pair of nodes $x, y \in K$ is connected by at least an operational path, independently of its length. Both the classical reliability and the *DCR* can be computed by application of a theorem of Moskowitz [14], also refered as the *Factoring Theorem*, in which the reliability of the probabilistic graph is computed in term of the reliabilities of two smaller graphs using a specific edge as pivot.

In many real-life situations the quality of the communication depends on the existence of a path connecting each pair of terminals x and y, whose length (measured as the number of edges) is bounded by a given integer D. The K-terminal Diameter-Constrained Network reliability is the probability of this event and it was originally introduced by Petingi and Rodriguez in 2001 [17] (for a survey on this reliability model refer to [16]). The *DCR* can be applied to assess performance objectives of for example packet-oriented networks where links may fail and there is a "time-to-live" (TTL) limit, specified in number of hops that can be traversed by any given packet (for instance IPv6 packets include a hop limit field [11]). It is also the case of many overlay networks (such as peer-to-peer file sharing networks) that employ flooding protocols for peer discovery which specify a maximum number of hops to be visited by a request (for instance Gnutella, which employs a flooding-based routing algorithm with a TTL value of 7 hops [19]). As the classical reliability measure does not capture the distances between the network nodes, the *DCR* can be applied to assess the probability of establishing a connection by setting, for example, the diameter bound D equal to the maximum number of hops to be visited by a packet or request. Another scenario for P2P networks is obtained if the link reliability value represents the probability that a pair of given nodes are in each other routing tables. In that case, the *DCR* models the fraction of the peers than can be reached from an arbitrary node. the The *DCR* measure subsumes the classical reliability in the following sense; as any path in a network on n vertices is composed of at most $n-1$ edges, then $R_K(G, D) = R_K(G)$, whenever $D = n - 1$. As the calculation of the classical reliability for an arbitrary set of terminal vertices K is an NP-Hard problem [1], then evaluation of the *DCR* is an NP-Hard problem as well. For fixed number of terminal vertices K, and for fixed values of the bound D, Cancela and Petingi [7] proved that to determine $R_K(G, D)$ is also an NP-Hard problem. Monte Carlo techniques have been shown to be excellent candidates to accurately estimate the classical reliability [4, 5] as well as to calculate the *DCR* [6].

For the classical reliability, topological reductions (e.g., series reduction, degree-2 reduction, parallel reduction, polygon-to-chain reductions) can be applied to a graph to reduce its size while preserving the reliability, and therefore reducing the computational effort for its evaluation. A series-parallel graph is a graph $G = (V, E)$ that can be reduced to single edge by application of series-parallel reductions, and, for any arbitrary set of terminal vertices K, $R_K(G)$ can be computed in time $O(|E|)$ [21]. For fixed values of the diameter bound D, with the exception of parallel reductions, the aforementioned transformations cannot be applied to evaluate the *DCR*, as these topological transformations may change the distance between nodes of a graph (and therefore its diameter). As for this case these transformations are not reliability-preserving, the deletion of D-irrelevant edges can be considered as the alternative to efficiently calculate $R_K(G, D)$, as well as possibly determine families of graphs in which $R_K(G, D)$ can be calculated in polynomial time.

An edge e of a graph G is said to be irrelevant if deletion of the edge (denoted as $G - e$) preserves the K-terminal Diameter-Constrained reliability, that is, $R_K(G, D) = R_K(G - e, D)$, for a given diameter bound D. For the specific case when $K = \{s, t\}$, an edge e that does not belong to any D-s,t-path can be then deleted without affecting the reliability. In [8] a preliminary study addressing D-irrelevancy was presented to show how the computational effort to evaluate the Source-to-terminal Diameter-Constrained reliability of graphs can be improved when irrelevant edges are efficiently detected (and then deleted); among other results we are extending this study by presenting new sufficient conditions to identify a superset of the D-irrelevant edges identified in [8].

The paper is structured as follows. In Section 2 we introduce notation and definitions pertaining to the concept of edge-irrelevancy and of the *DCR*. As the classical reliability is a special case of the Diameter Constrained reliability (i.e., when the diameter bound $D = n - 1$ for graphs on n vertices), in Section 3 we present a review of known reliability-preserving topological transformations applied to improve the computational effort to evaluate the classical reliability (and therefore of the *DCR*), and characterize families of graphs for which the reliability can be computed in polynomial time. Since for fixed values of D, these transformations are not reliability-preserving, we introduce new sufficient conditions, in addition to the ones presented in [8], to efficiently identify irrelevant edges in graphs. In Section 4, we show how these sufficient conditions can be embedded within the frame of a procedure based on Moskowitz's Theorem (i.e., Factoring Theorem) to evaluate $R_{\{s,t\}}(G, D)$. In Section 5 we present a computational analysis of Factoring Theorem, when a procedure to detect irrelevant edges is present or not, in order to compute $R_{\{s,t\}}(G, D)$, based on computational experiments (illustrated in Appendix A) performed on different families of topologies. Finally, in Section 6, we present conclusions and further research.

2 Definitions and notation

In this section we introduce definitions and notation that will be used in the sequel.

Definition 1 *(Path) - An s,t-path is a sequence of distinct vertices $< u_0 = s, u_1, u_2, \ldots, u_{r-1}, t = u_r >$ of a graph G, such that (u_i, u_{i+1}) is an edge of G, $0 \leq i \leq r-1$. If each edge $e = (u,v)$ is assigned an integer weight $w(e)$, the length of an s,t-path $p =< u_0 = s, u_1, u_2, \ldots, u_{r-1}, t = u_r >$ is $L_G(p) = \sum_{i=0}^{r-1} w((u_i, u_{i+1}))$. For the unweighted case, or equivalently when each edge is assigned a weight of one, then the length $L_G(p)$ is the number of edges comprising the path p.*

Definition 2 *(Simple Cycle) - A simple cycle is a sequence of vertices $< u_0 = x, u_1, u_2, \ldots, u_{r-1}, y = u_r >$ of a graph G, such that (u_i, u_{i+1}) is an edge of G, $0 \leq i \leq r-1$, and all the vertices of the sequence are distinct except for $x = y$.*

Definition 3 *(Irrelevancy and Critical graphs) - An s,t-path p is called a D-s,t-path if $L(p) \leq D$ (i.e., the path length is at most D). Given a graph $G = (V,E)$, a distinguished set of terminal vertices set $K = \{s,t\}$, and a diameter bound D, and edge $e = (u,v) \in E$ is said to be D-relevant if e lies in some D-s,t-path, otherwise e is D-irrelevant. If every edge of G is D-relevant then G is called D-diameter-critical.*

Definition 4 *(Distance and Diameter) - The distance between two vertices x,y of G is $distance_G(x,y) = min\{L_G(p) : p \text{ is an } x,y\text{-path of } G\}$. Moreover the K-diameter is the maximum distance between vertices of K.*

Definition 5 *(Degree) - The degree of a vertex v of a graph G, denoted as $deg_G(v)$, is the number of edges incident at v.*

We are also introducing notation to describe topological transformations in graphs.

- Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and two distinct vertices a, b such that $a \in V_1$ and $b \in V_2$, let $G_1 a.e.b G_2$ describe the operation of joining G_1 and G_2 by an edge $e = (a,b)$.
- Given two paths $p_1 =< u_0, u_1, u_2, \ldots, u_{r-1}, u_r >$, and $p_2 =< v_0, v_1, v_2, \ldots, v_{m-1}, v_m >$, and two distinct vertices u and v of p_1 and p_2, respectively, let $p_1 u.e.v p_2$ describe the operation of joining p_1 and p_2 by an edge $e = (u,v)$.

In the following section we present a review of well-known reliability-preserving transformations applicable to the classical reliability for a graph $G = (V,E)$ (and therefore applicable to evaluate the *DCR* when $D = |V| - 1$) and the D-irrelevancy problem to efficiently identify irrelevant edges for fixed values of D.

3 Reliability-perserving transformations for the classical and Diameter Constrained reliability measures

3.1 Reductions applicable to the classical reliability and to the DCR when $D = n-1$, for graphs on n vertices

This sub-section is intended as a brief review of topological reductions known to reduce the computational effort to evaluate the classical reliability (further detailed information can be found in [20, 21, 22, 24, 25]).

Given a probabilistic graph $G = (V,E)$, the K-terminal Diameter Constrained reliability, $R_K(G,D)$, is equal to the K-terminal reliability, $R_K(G)$, whenever $D = n-1$, where $n = |V|$; consequently reliability-preserving reductions applicable for the classical case to reduce the computational effort to evaluate $R_K(G)$, in addition to possibly identifying families of graphs in which the reliability can be evaluated in polynomial time (as the general problem is NP-Hard), can be extended for the evaluation of the *DCR* as well.

For fixed values of the diameter bound D, and for the case of the *DCR*, these reductions are not reliability-preserving, as they may change the distance between nodes of a graph in an arbitrary way; in the next sub-section we will look at D-irrelevancy as the alternative way to reduce the computation effort to evaluate $R_K(G,D)$, for fixed D.

A *simple reduction*, is a reliability-preserving transformation that reduces the size of a graph $G = (V,E)$, i.e. reduces $|V| + |E|$, and therefore the computational complexity of evaluating $R_K(G)$: Certain edges and/or vertices in G are replaced to obtain G'; possibly a new set K' is defined; and a multiplicative factor Ω is defined; all such that $R_K(G) = \Omega R_{K'}(G')$.

The following are simple reliability-preserving reductions [20]:

- A *parallel* reduction replaces a pair of edges $e_1 = (u,v)$ and $e_2 = (u,v)$, with a single edge $e_3 = (u,v)$ with reliability $r(e_3) = 1 - q(e_1)q(e_2)$, $K' = K$, and $\Omega = 1$.

- ◦ A *series* reduction replaces two edges $e_1 = (u, v)$ and $e_2 = (v, w)$, under the assumption that $deg_G(v) = 2$ (see Definition 5) , with a single edge $e_3 = (u, w)$ with reliability $r(e_3) = r(e_1)r(e_2)$, $K' = K$, and $\Omega = 1$.
- ◦ A *degree-2* reduction replaces two edges $e_1 = (u, v)$ and $e_2 = (v, w)$, under the assumption that $deg_G(v) = 2$, and $\{u, v, w\} \subseteq K$, with a single edge $e_3 = (u, w)$ with reliability $r(e_3) = r(e_1)r(e_2)/(1 - q(e_1)q(e_2))$, $K' = K - \{v\}$, and $\Omega = 1 - q(e_1)q(e_2)$.

A graph $G = (V, E)$ is called *series-parallel* if it can be reduced to a single edge by successive applications of series-parallel reductions mentioned above. If a probabilistic series-parallel graph $G = (V, E)$, with terminal vertex set K, can be reduced to a single edge, then this graph is called *series-parallel-reducible* and $R_K(G)$ can be computed in polynomial time. However, depending upon the location of the terminal vertices, there are other series-parallel graphs that cannot be reduced to a single edge (*series-parallel-irreducible*) by successive applications of these simple reductions.

In [21], Satyanarayana and Wood introduced a new set of reliability-preserving non-simple topological transformations, called *polygon-to-chain* reductions, which in combination with the series-parallel and degree-2 reductions aforementioned, reduce any probabilistic series-parallel graph $G = (V, E)$, and for any arbitrary set of terminal vertices $K \subseteq V$, into a single edge in time $O(|E|)$; consequently the evaluation of the classical reliability for any series-parallel graph is achievable in polynomial time.

A *chain* ξ of a graph G is an alternating sequence of vertices and edges $u_0, (u_0, u_1), u_1, (u_1, u_2), u_2, \ldots, u_{r-1}, (u_{r-1}, u_r), u_r$ such that the internal vertices $u_1, u_2, \ldots, u_{r-1}$ of the chain have all degree-2 in G and u_0, and u_r have degree greater than 2. The length of a chain is simply the number of edges it contains. Suppose next that ξ_1 and ξ_2 are two distinct chains of length l_1 and l_2 respectively; if the two chains have common end-vertices u and v in G, then $\Delta = \xi_1 \cup \xi_2$ forms a *polygon*. Satyanarayana and Wood showed that any polygon, independently of its length, can be transformed, using the simple reductions, into seven elementary polygons in which terminal vertices were assigned to different vertices of these polygons. By a complex analysis, they showed that these elementary polygons could be always transformed into chains while preserving the calculation of the reliability.

For fixed values of the diameter bound D, except for parallel reductions, the aforementioned reductions are not reliability-preserving in case of the *DCR*. In the next sub-section we consider how deletion of D-irrelevant edges could be considered as the alternative to minimize the computational effort to evaluate $R_K(G, D)$, for constant values of D and when $K = \{s, t\}$.

3.2 The problem of D-irrelevancy and efficient calculation of the Source-to-terminal Diameter Constrained reliability

Given an edge e and two vertices s and t of a graph G, e is D-relevant if and only if there exist an s, t-path p among all s, t-paths containing e, such that $L_G(p) \leq D$. Thus we are considering first the optimization problem of finding a shortest s, t-path of G containing an specific edge e.

For the case in which negative integer weights can be assigned to the edges of the graph, we show next that to find a shortest s, t-path containing an specific edge e is NP-Hard, by transforming the *Longest Path* problem in which positive integer weights are assigned to the edges of a graph, into this decision problem:

P1: *Shortest Path Containing an Specific Edge (SPE)*

Instance: Graph $G' = (V', E')$, edge e^*, vertices $s', t' \in V'$, length $l(e') \in \mathcal{Z}$, for each $e' \in E'$, and bound $k' \in \mathcal{Z}$.

Question: Is there an s', t'-path p' of length k' or less containing the edge e^* (i.e., $L(p') \leq k'$)?

Consider next the Longest Path problem, to be known to belong to the NP-Complete computational class, when positive integer weights are assigned to the edges [13]:

P2: *Longest Path (SPP)*

Instance: Graph $G = (V, E)$, vertices $s, t \in V$, length $l(e) \in \mathcal{Z}^+$, for each $e \in E$, and bound $k \in \mathcal{Z}^+$.

Question: Is there an s, t-path p of length k or greater (i.e., $L(p) \geq k$)?

Lemma 6 *The Shortest Path Containing an Specific Edge problem,* **P1***, is NP-Hard.*

Proof: It is not known if SPE is in NP. Consider the transformation from the Longest Path problem to the *SPE* problem.

Let $G = (V, E)$, vertices $s, t \in V$, length $l(e) \in \mathcal{Z}^+$, bound $k \in \mathcal{Z}^+$ be an instance of **P2**. Let $G' = (V', E')$ be graph obtained by joining two copies of G by an edge $e^* = (a, b)$ where a is vertex t of the first

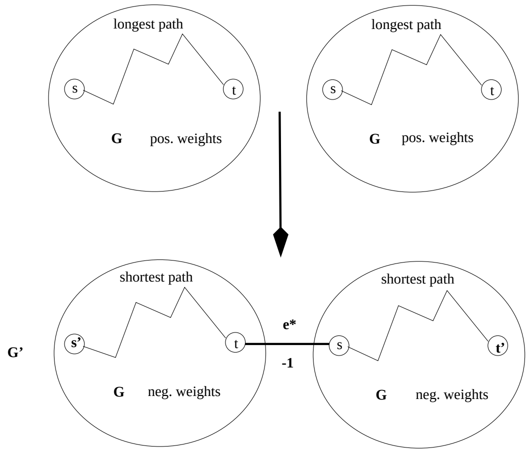

Figure 1: Transformation from the Longest Path problem to the Shortest Path Containing an Specific Edge.

copy of G and b is vertex s of the second copy of G, that is, $G' = Gt.e^*.sG$. Moreover let the vertices s' and t' be the vertices of G' corresponding to vertex s of first copy of G and vertex t of the second copy of G, respectively. In addition, for each edge $e' \in E'$, let $l(e') = -l(e)$ where e is e' corresponding edge in G (as the edges of G have been assigned positive weights, then the edges of G' have negative weights). Moreover let $l(e^*) = -1$, and $k' = -2k - 1$ (see Figure 1).

Suppose G contains an s,t-path p of length $L_G(p) \geq k$. As $L_{G'}(p) = -L_G(p)$ then the path $p' = pt.e^*.sp$ is an s',t'-path in G' containing e^*. As $L_G(p) \geq k$ then $L_{G'}(p') \leq -2k - 1$.

Conversely let p' be an s',t'-path in G' of length $L_{G'}(p') \leq k'$. The path p' must contain the edge e^*, as otherwise p' is not a s',t'-path, thus $p' = p'_1t.e^*.sp'_2$, where p'_1 and p'_2 are s,t-paths in the first and second copy of G, respectively. But as $L_{G'}(p') \leq k' = -(2k+1)$ then either $L_{G'}(p'_1) \leq -k$ or $L_{G'}(p'_2) \leq -k$, then G contains a path whose length is greater or equal to k, as p'_1 and p'_2 are paths in G □

If the optimization problem *SPE* is extended to find a shortest s,t-path containing a predefined set of edges (or vertices), then the problem is NP-Hard as well, as the above proof can be generalized by replacing the edge e^* by a path with a fixed number of edges, where each edge of the path is assigned a weight of -1.

When positive weights are assigned to the edges, little is known regarding the corresponding optimization problem of finding a shortest s,t-path containing an specific edge (or a predefined set of edges), even though this problem has been studied since the 1960s. In a recent paper by Bijörklund et al. [2], the authors studied the problem of finding a shortest simple cycle through a set of elements $B \subset V \cup E$ (i.e., vertices and edges), and a randomized algorithm (of order $2^{|B|}n^{O(1)}$) was introduced for any arbitrary set B of elements. This problem is equivalent to the problem of finding a shortest s,t-path through a set of edges (or vertices), as an edge between s and t can be added, if s and t are not adjacent, to form a simple cycle. However, according to this paper, little is known regarding the optimal solution of this optimization problem when $b = |B|$ is a constant, $b \geq 3$. Thus it is not clear if there exist a computationally efficient way to determine necessary and sufficient conditions to classify an edge e as D-irrelevant or not. However in [8], sufficient conditions to identify D-irrelevant edges were introduced, and are stated in the following proposition:

Proposition 7 *Given a graph $G = (V, E)$, a diameter bound $D \in \mathcal{Z}^+$, and an edge $e = (u, v)$ of G. If $distance_G(s, u) + distance_G(v, t) \geq D$ and $distance_G(s, v) + distance_G(u, t) \geq D$, then e is D-irrelevant.*

Given a graph $G = (V, E)$, let $n = |V|$ and $m = |E|$. The distance (see Definition 4) between s and any other vertex of the graph, can be efficiently determined by application of Dijkstra's Shortest Path algorithm (**DSPA** for short) [12] of order $O(m + n)$. From Definition 3, it follows that edges that do not belong to any s,t-path are also irrelevant; it is important to note that if G is one-connected (i.e., there exist a vertex, called a *cut-point*, whose deletion disconnects G), and s and t belong to a two-connected component $C = (V_c, E_c)$, any edge e that does not belong to this component, is also D-irrelevant as no s,t-path containing e exists in G; thus it is possible that Proposition 7 won't recognize e as irrelevant. In this case, edges that belong to blocks (two-connected components) other than C, can be efficiently identified by a linear-time algorithm based on bi-connectivity theory [10]. Next suppose that G has more than one connected component, then if s and t belong to a same connected component (not necessarily a two-connected one) $C = (V_c, E_c)$, any edge e that does not belong to C is also irrelevant. In this case Proposition 7 will classify this edge as D-irrelevant as the original distance between s and a end-point of e is set to infinite by Dijkstra's algorithm.

We are proposing next sufficient conditions which detect a superset of the set of D-irrelevant edges identified by Proposition 7.

Proposition 8 *Given a graph $G = (V, E)$, a diameter bound $D \in \mathcal{Z}^+$, and an edge $e = (u, v)$ of G. If $distance_{G-e}(s, u) + distance_{G-e}(v, t) \geq D$ and*

$distance_{G-e}(s,v) + distance_{G-e}(u,t) \geq D$, *then* e *is* D*-irrelevant.*

Proof: Suppose that $e = (u,v)$ is D-relevant, then as e belongs to a D-s,t-path, at least one of the following conditions is true:

1. There exist an s,t-path $p =< s = u_1, u_2, \ldots, u, v, \ldots, u_L, u_{L+1} = t >$ of length $L \leq D$. Thus the length of sub-path $p_1 =< s = u_1, u_2, \ldots, u >$ plus the length of sub-path $p_2 =< v, \ldots, u_L, u_{L+1} = t >$ is at most $D-1$.

2. There exist an s,t-path $p =< s = v_1, v_2, \ldots, v, u, \ldots, v_{L'}, v_{L'+1} = t >$ of length $L' \leq D$. Thus the length of sub-path $p_3 =< s = v_1, v_2, \ldots, v >$ plus the length of sub-path $p_4 =< u, \ldots, v_{L'}, v_{L'+1} = t >$ is at most $D-1$.

As the sub-paths p_1, p_2, p_3, and p_4 are paths in $G-e$, then

- $distance_{G-e}(s,u) + distance_{G-e}(v,t) < D$, or
- $distance_{G-e}(s,v) + distance_{G-e}(u,t) < D$.

By the contrapositive, if $(distance_{G-e}(s,u) + distance_{G-e}(v,t) \geq D)$ and $(distance_{G-e}(s,v) + distance_{G-e}(u,t) \geq D)$ then e is D-irrelevant □

It is obvious that if the conditions stated in Proposition 7 recognize a set of D-irrelevant edges S_1, then the conditions specified by Proposition 8 find a set of D-irrelevant edges S_2 where $S_1 \subseteq S_2$.

In Figure 2 a graph is depicted in which edges $(1,2), (2,3)$, and $(3,4)$ are D-irrelevant, whenever $D = 5$. In this case $S_1 = \emptyset$ as the conditions stated by Proposition 7 are not met. However by application of Proposition 8, if $e = (1,2)$, $(distance_{G-e}(s,1) + distance_{G-e}(2,t) \geq 5)$ and $(distance_{G-e}(s,2) + distance_{G-e}(1,t) \geq 5)$, as $distance_{G-e}(s,1) = 1$, $distance_{G-e}(2,t) = 4$, $distance_{G-e}(s,2) = 4$, and $distance_{G-e}(1,t) = 4$. Similarly edges $(2,3)$, and $(3,4)$ are also identified as D-irrelevant by Proposition 8.

We next present two procedures, **Irrelevant-**$P_1()$ ([8]) and **Irrelevant-**$P_2()$, to detect irrelevant edges, based on Proposition 7 and Proposition 8, respectively:

Irrelevant-$P_1()$
Input: Graph $G = (V,E)$*, terminal set* $K = \{s,t\}$*, diameter bound* D*.*

1. *delete from* G *edges that do not belong to any* s,t*-path.*

2. *call* **DSPA** *twice, once to find the* $distance_G(s,v)$ *for every vertex* $v \in V$*, and then to find* $distance_G(t,v)$ *for every vertex* $v \in V$ *(order* $O(m+n)$*).*

3. *delete the edges from G that meet the conditions stated in Proposition 7.*

4. **if** *there still edges in G that do not belong to any* s,t*-path, go to* **Step 1***,*
   else Exit.

Irrelevant-$P_2()$
Input: Graph $G = (V,E)$*, terminal set* $K = \{s,t\}$*, diameter bound* D*.*

1. *delete from* G *edges that do not belong to any* s,t*-path.*

2. for every edge $e = (u,v)$ of G do

   2.1 *delete e from G. Apply* **DSPA** *to* $G-e$*.*

   2.2 **if** *conditions stated in Proposition 8 for the end-vertices of* $e = (u,v)$ *are met* **Skip** *(the edge is* D*-irrelevant thus consider* $G-e$*),*
   else *put-back e in G.*

3. **if** *there still edges in G that do not belong to any* s,t*-path, go to* **Step 1***,*
   else Exit.

In Step 1 of procedures **Irrelevant-**$P_1()$ and **Irrelevant-**$P_2()$, all the edges that do not belong to any s,t-path can be identified by a linear-time algorithm based on bi-connectivity theory [10].

Despite of the fact that procedure **Irrelevant-**$P_2()$ finds a superset of the edges detected by **Irrelevant-**$P_1()$, the conditions stated in Step 2 of the latest are determined by just two application of Dijkstra's algorithm in time $O(m+n)$, while the distance conditions stated in Proposition 8 are determined in time $O(m^2)$, as we must apply **DSPA** m times (i.e. Step 2 of **Irrelevant-**$P_2()$), each time when we delete a possible irrelevant edge e from G; the trade-off between the number of irrelevant edges recognized, and the computational complexity for detecting these edges, when applying these procedures, has to be further investigated (Section 5).

As it was mentioned in the Introduction, both the classical reliability and the *DCR* can be computed by application of a procedure based on a theorem of Moskowitz (i.e., Factoring Theorem), in which the reliability of the probabilistic graph G can be computed in term of the reliabilities of two (possibly smaller) networks derived from G by fixing the state of a selected edge e either up (i.e., e is operational) or down (i.e., e failed). In the next section we consider the

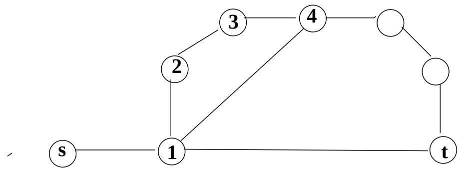

Figure 2: Graph $G = (V, E)$ and diameter bound $D = 5$.

problem of embedding a procedure to determine irrelevant edges (by either applying **Irrelevant-**$P_1()$ or **Irrelevant-**$P_2()$) within the context of Moskowitz's Theorem.

4 Moskowitz and computation of the Source-to-terminal Diameter-Constrained Reliability

Moskowitz's Decomposition Theorem express the reliability of a network G as a function of the reliabilities of the two networks obtained from G by fixing the state of a selected edge e either up (i.e., r_e is set to 1) or down (i.e., r_e is set to 0). Moskowitz's decomposition was extensively used within the context of the classical reliability (see [15, 18, 22, 24, 25]). Within this context we say that the random state of an edge e is *undetermined* if $0 < r_e < 1$ [8].

Theorem 9 *For any network G that has at least one edge e whose random state is undetermined then*

$$R_{\{s,t\}}(G, D) = r_e R_{\{s,t\}}(G * e, D) + (1 - r_e) R_{\{s,t\}}(G - e, D), \text{ where}$$

- *e is an edge with undetermined random state in G if $0 < r_e < 1$.*
- *$G * e$ is the network obtained from G by fixing the edge e up (i.e., $r_e = 1$).*
- *$G - e$ is the network obtained from G by fixing edge e down (i.e., $r_e = 0$, or equivalently e is deleted from G).*

Consider a procedure **Factoring( )** to evaluate $R_{\{s,t\}}(G, D)$, derived from Theorem 9; this procedure describes a binary tree in which each node j of this tree represents a subgraph of G, G_j, in which its edges are either operational (reliability 1), have failed (reliability 0), or whose random states are undetermined (the root node of the derived binary tree represents the original network G). For each of the possible subgraphs G_j's, its Source-to-terminal Diameter-Constrained reliability is then calculated as:

$$R_{\{s,t\}}(G_j, D) = \begin{cases} 0: \text{ if there is no } D\text{-}s,t\text{-path in } G_j. \\ 1: \text{ if } G_j \text{ contains an operational } D\text{-}s,t\text{-path.} \\ r_e R_{\{s,t\}}(G_j * e, D) + (1 - r_e) R_{\{s,t\}}(G_j - e, D): e \text{ is undetermined.} \end{cases}$$

We can embed either procedure **Irrelevant-**$P_1()$ or procedure **Irrelevant-**$P_2()$, described in Section 3, within **Factoring( )**, to possibly delete irrelevant edges in each of the states G_j of the binary tree generated by the application of the recursive function previously stated, to possibly shorten the computational effort. The following procedure, called **Fact-Reductions**$-i()$, evaluates the Source-to-terminal Diameter-Constrained reliability of a network while enforcing the deletion of irrelevant edges identified by procedure **Irrelevant-**$P_i()$ ($i \in \{1, 2\}$). Procedure **Fact-Reductions**$-i()$ receives five parameters, namely the network topology G, the source and terminal nodes s and t, the diameter constraint D, and a flag called $flagr$, which indicates whether further reductions are or are not possible. At the first invocation, $flagr$ is set to 1.

Procedure **Fact-Reduction**$-i(G, s, t, D, flagr)$
Input: network $G = (V, E)$, s, t, D, and $flagr$
Output: reliability $R_{\{s,t\}}(G, D)$

1. Check end recursion condition:
   1.1. If G contains a D-s,t-path having only operational edges return (1).
   1.2. If there is no D-s,t-path in G return (0).
2. Apply procedure to detect irrelevant edges:
   2.1. If $(flagr = 1)$ *call* **Irrelevant-**$P_i(G, s, t, D)$.
3. Select randomly an edge e in G with undetermined state.
4. Solve recursively for $G - e$: $R_{\{s,t\}}(G - e, D)$ = **Fact-Reduction**$-i(G - e, s, t, 1)$.
5. Solve recursively for $G * e$: $R_{\{s,t\}}(G * e, D)$ = **Fact-Reduction**$-i(G * e, s, t, 0)$.
6. Compute $R_{\{s,t\}}(G, D)$: return ($R_{\{s,t\}}(G, D) = (1 - r_e) R_{\{s,t\}}(G - e, D) + r_e R_{\{s,t\}}(G * e, D)$).

In the next section we present experimental results to compare the computational performances of **Factoring( )**, **Fact-Reduction**$-1()$, and **Fact-Reduction**$-2()$.

5 Analysis of experimental results illustrated in Appendix A

In this section we compare the computational effort and other performance metrics yielded by the three methods mentioned in Section 4, that is, **Factoring( )**, and **Fact-Reduction**$-1()$, **Fact-Reduction**$-2()$, based on experimental results illustrated in Appendix A. Tests were performed on different families of topologies, which encompass the Arpanet, the 5X5-Grid, the Dodecahedron, and Circulant graphs $C^n_{1,n/2}$ on n vertices with jumps 1 and $n/2$ (see Figure 3), and they were chosen on the basis of they previous use in the related literature. As most of these topologies are of relatively low-density, we have also performed tests on high-density graphs such as complete graphs (K_n) on n vertices.

In [8] computational experiments were performed on the same families of topologies as the one shown in Figure 3, to compare the performance of **Factoring( )**, with the performance of **Fact-Reduction**$-1()$, and by taking into account different values of the diameter bound D. The tests performed in this work were geared to complement the study presented in [8], by also presenting computational results obtained by embedding procedure **Irrelevant-**$P_2()$ within **Factoring( )**, and run on the same set of topologies.

For all tests, an unique probability of failure $q_e = 0.5$ was assigned to each edge $e \in E$. In Table 1 and Table 2 of Appendix A, the data shown in columns 1 through 3 represent the type of topology, the label of the source and terminal nodes of the topology, and the diameter bound D, respectively. Column 4 shows the value of the Source-to-terminal Diameter Constrained reliability; columns 5 and 6, 7 and 8, and 9 and 10, represent the tree size (i.e., the number of nodes generated by factoring recursive algorithm) and CPU time (in seconds) taken by **Factoring( )**, **Fact-Reduction**$-1()$, and **Fact-Reduction**$-2()$ procedures, respectively.

In agreement with the conclusions stated in [8], the experimental results illustrated by Table 1 and Table 2 show a consistent computational gain observed when elimination of irrelevant edges was performed on most topologies, either by applying **Fact-Reduction**$-1()$ or **Fact-Reduction**$-2()$, especially when low-density topologies were tested (e.g., Circulants, Dedocahedron); the most significant gain was observed when tests were performed on the 5X5-Grid topology (Table 1). We think that detection of irrelevant edges may play an important role when studying the computational complexity for evaluating $R_{\{s,t\}}(G, D)$ for low-density graphs.

For topologies composed of n nodes, the computational gain when detecting irrelevant edges is particularly important for low values of the diameter bound D, and it becomes less significant when D increases toward the maximum value $n-1$ (i.e., as stated in the Introduction $R_{\{s,t\}}(G, D)$ approaches the classical reliability value $R_{\{s,t\}}(G)$), or when tests were performed on high-density topologies (e.g., K_7 or K_8).

The comparison between **Fact-Reduction**$-1()$ and **Fact-Reduction**$-2()$ yields a computational gain when the latest was applied on most of the classes of topologies, and for most values of the diameter bound D, except when the D bound was approaching $n-1$, or when tests were conducted on high-density graphs. It is important to note that the number of recursive calls (i.e., tree size) was consistently less when **Fact-Reduction**$-2()$ was applied on most topologies.

6 Conclusions and further research

The purpose of this work is to present a study of the optimization problem of identifying D-irrelevant edges and diameter-related problems in graphs, from a computational point of view. Since presently is unknown if necessary and sufficient conditions that can efficiently (polynomial time) detect all irrelevant edges exist (the equivalent problem when edges with negative integer weights are allowed is NP-Hard), we have introduced new sufficient conditions that can efficiently recognize a subset of the D-irrelevant edges. We then presented numerical results illustrating the importance of embedding a procedure to detect irrelevant edges based on these conditions, in combination with an algorithm derived from Moskowitz's Decomposition Theorem, in order to evaluate $R_{\{s,t\}}(G, D)$.

Future work will comprise the determination of classes of topologies for which $R_{\{s,t\}}(G, D)$ can be evaluated in polynomial time (e.g., sparse graphs) as suggested by the computational analysis presented in Section 5.

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7 Appendix A (experimetal results)

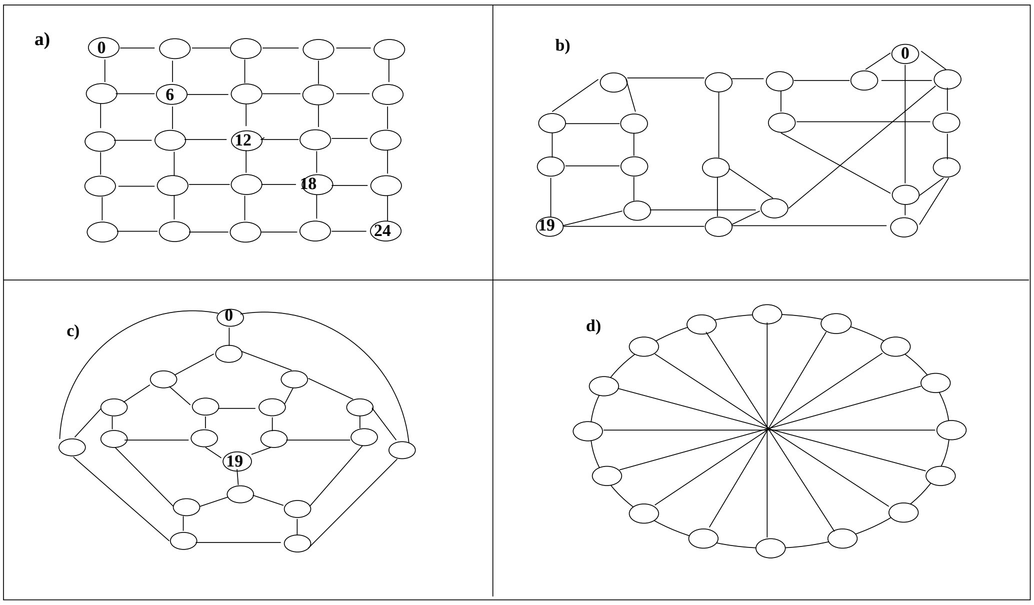

Figure 3: Classes of topologies: a) 5X5-Grid, b) Arpanet, c) Dodecahedron, and d) Circulant $C^{16}_{1,8}$.

G	(s,t)	D		*Fact*		*Fact-R-1*		*Fact-R-2*	
			$R_{s,t}(G,D)$	tree-size	CPU-t (s)	tree-size	CPU-t (s)	tree-size	CPU-t (s)
5X5Grid	(0,20)	8	0.176359	N/A	$>$ 24 hours	317631	1	232109	1
5X5Grid	(0,6)	8	0.498971	353362807	883	43661	<1	28723	<1
5X5Grid	(0,12)	8	0.339515	1.09e +09	2706	16333687	98	8559269	93
5X5Grid	(0,18)	8	0.234521	N/A	$>$ 24 hours	1.005e +09	4903	483325443	4113
5X5Grid	(0,24)	8	0.123324	N/A	$>$ 24 hours	1.035e +09	5246	1.035e +09	7685

Table 1: Comparison of the three methods (**Factoring()**, **Fact-Reduction**$-1()$, and **Fact-Reduction**$-2()$) on the 5X5-Grid. Computations whose execution time exceeded 24 hours were aborted.

G	(s,t)	D		*Fact*		*Fact-R-1*		*Fact-R-2*	
			$R_{s,t}(G,D)$	tree-size	CPU-t (s)	tree-size	CPU-t (s)	tree-size	CPU-t (s)
Dodeca	(0,19)	5	0.168441	44015523	95	27935	<1	27935	< 1
		7	0.268820	79223841	170	19313039	89	11353599	79
		9	0.285391	92836783	201	66597743	298	31235671	255
Arpanet	(0,19)	4	0.162109	13803603	19	97	< 1	97	< 1
		6	0.237618	62130037	103	362293	1	80109	< 1
		9	0.295711	214988177	373	106920719	399	61103239	371
		19	0.302415	263588889	475	261395215	876	109031949	690
K_6	(0,5)	3	0.913330	44673	< 1	32305	< 1	29731	< 1
		5	0.923584	45417	< 1	45369	< 1	42655	< 1
K_7	(0,6)	3	0.953217	2634387	9	1535477	22	1454955	23
		4	0.962086	2675467	8	2593747	40	2482975	34
		5	0.963001	2679667	8	2673787	38	2566875	46
K_8	(0,7)	2	0.911011	277526209	1550	2915	0	2915	0
		3	0.975735	313904521	1185	140021817	1143	135421811	1343
		5	0.982520	317848021	1197	317225461	2557	308845439	3358
		7	0.982573	317878981	234	317877541	546	309523559	678
$C^{20}_{1,10}$	(0,19)	5	0.578125	207057	0	337	0	157	0
		9	0.581543	1592451	2	7013	0	2813	0
		11	0.584226	6190747	11	1264923	4	1185547	4
		17	0.587613	15617515	27	15188151	48	6622651	33

Table 2: Comparison of the three methods (**Factoring()**, **Fact-Reduction**$-1()$, and **Fact-Reduction**$-2()$) on the Dodecahedron, Arpanet, Complete graphs on 6, 7, 8 vertices, and the Circulant on 20 vertices with jumps 1 and 10.